Geometry of complex Monge-Ampère equations on compact Kähler manifolds

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GEOMETRY OF COMPLEX MONGE-AMPÈRE EQUATIONS ON COMPACT KÄHLER MANIFOLDS

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Contents

Introduction

In the mid 70's, Aubin-Yau [Aub76, Yau78] solved the problem of the existence of Kähler metrics with constant negative or identically zero Ricci curvature on compact Kähler manifolds. In particular, they proved the existence and regularity of the solution of the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\omega^n$$

where the reference form ω is Kähler and the density f is smooth.

In this thesis we look at degenerate complex Monge-Ampère equations, where the word "degenerate" stands for the fact that the reference class is merely big and not Kähler or that the densities have some divisorial singularities.

When looking at an equation of the type

$$(\theta + dd^c\varphi)^n = \mu \qquad (\star)$$

where μ is a positive measure, it is not always possible to make sense of the left-hand side of ($\star$). It was nevertheless observed in [GZ07] and [BEGZ10] that a construction going back to Bedford and Taylor enables in this global setting to define the non-pluripolar part of the would-be positive measure $(\theta + dd^c\varphi)^n$ for an arbitrary θ-psh function, where θ represents a big class.

The notion of big classes is invariant by bimeromorphism while this is not the case in the Kähler setting. It is therefore natural to study the invariance property of the non-pluripolar product in the wider context of big cohomology classes. We indeed show that it is a bimeromorphic invariant.

Generalizing the Aubin-Mabuchi energy functional (cf. [Aub84, Mab86] and [BB10] for the extension to the singular setting), in [BEGZ10] the authors introduced weighted energies associated to big cohomology classes. Under some natural assumptions, we show that such energies are also bimeromorphic invariants.

We also investigate probability measures with finite energy (this concept was introduced in [BBGZ13]) and we show that this notion is a biholomorphic but not a bimeromorphic invariant. Furtheremore, we give criteria insuring that a given measure has finite energy and test these on various examples.

We then study complex Monge-Ampère equations on quasi-projective varieties. In particular we consider a compact Kähler manifold X, $D \subset X$ a divisor and we look at the equation

$$(\omega + dd^c\varphi)^n = f\omega^n$$

where f is smooth outside D and with a precise behavior near the divisor. We prove that the unique normalized solution φ is smooth outside D and we are able to describe its asymptotic behavior near D (joint work with Hoang Chinh Lu). The solution is clearly not bounded in general and thus the idea is to find a convenient "model" function (a priori singular) bounding from below the solution. To do so we introduce generalized Monge-Ampère capacities, and use them following Kołodziej's approach [Koł98] who deals with globally bounded potentials.

These capacities, which generalize the Bedford-Taylor Monge-Ampère capacity, turn out to be the key point when investigating the existence and the regularity of solutions of complex Monge-Ampère equations of type

$$\mathrm{MA}\,(\varphi) = e^{\lambda\varphi} f\omega^n, \qquad \lambda \in \mathbb{R}$$

where f has divisorial singularities.

We also treat some cases when f is not in L^1, an important issue for the existence of singular Kähler-Einstein metrics on general type varieties with log-canonical singularities [BG13].

Chapter 1

Preliminaries and presentation of the results

1.1 Big cohomology classes

1.1.1 Positive Currents

Consider a real oriented manifold M of dimension m. Recall that a *current* T of dimension q (or degree $m-q$) on M is a continuous linear form on the vector space $\mathcal{D}^q(X)$ of smooth differential forms of degree q with compact support. We denote by $\mathcal{D}'_q(M)$ (or ${\mathcal{D}'}^{m-q}(M)$) the space of currents of dimension q on X, and by $\langle T, u\rangle$ the pairing between a test q-form u and a current T of dimension q. A first example of a current of dimension q is the *current of integration* over a closed oriented submanifold Z of dimension q and of class C^1, which is denoted by $[Z]$ and defined as

$$\langle [Z], u\rangle := \int_Z u.$$

Observe that, given f a q-form with coefficients in $L^1_{loc}(M)$, we can associate the current T_f of dimension $m-q$ (and degree q) defined as follows:

$$\langle T_f, u\rangle := \int_M f \wedge u.$$

Given a current T of degree q, the wedge product of T with a smooth p-form v is defined as

$$\langle T \wedge v, u\rangle := \langle T, v \wedge u\rangle.$$

One can also define the exterior derivative dT as the $(q+1)$-current satisfying

$$\langle dT, u\rangle := (-1)^{q+1}\langle T, du\rangle.$$

A current T is then said to be *closed* if $dT = 0$. We denote by $\{T\}$ the cohomology class defined by the current T. By deRham's Theorem, the corresponding cohomology vector space

$$H^{m-q}(M) := \{\text{closed currents of degree}\, q\}/\{dS \mid S\, \text{current of degree}\, q-1\}$$

is isomorphic to the one defined using closed smooth differential forms.

Let now X be a complex manifold of complex dimension n. The decomposition of complex valued differential forms according to their bidegrees induces a decomposition at the level of currents. We say that a current T is of bidegree (p,q) if it is of degree $p+q$ and $\langle T, u\rangle = 0$ for any test form u of bidegree $(k,l) \neq (n-p, n-q)$. We denote by $\mathcal{D}'^{p,q}(X)$ the space of such currents, and by $H^{p,q}(X)$ the corresponding vector space of cohomology classes.

In the complex case one can define a notion of positivity at the level of forms and currents. Let V be a complex vector space of dimension n and $(z_1, ..., z_n)$ coordinates on V. Observe that V has a canonical orientation defined by the volume form

$$(idz_1 \wedge d\bar{z}_1) \wedge ... \wedge (idz_n \wedge d\bar{z}_n)$$

and a (n,n)-form on V is said to be positive if and only if it is a positive multiple of the orientation form. A (p,p)-form u is said to be *positive* if for all $\alpha_j \in V^*$, $1 \leq j \leq n-p$, we have that

$$u \wedge (i\alpha_1 \wedge \bar{\alpha_1}) \wedge ... \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$$

is a positive (n,n)-form. Equivalently, a form of bidegree (p,p) is positive if and only if its restriction to every p-dimensional subspace $S \subset V$ is a positive volume form on S.

The set of positive (p,p)-forms is a closed convex cone in $\bigwedge^{p,p} V^*$ and its dual cone in $\bigwedge^{n-p,n-p} V^*$ is the *strongly positive cone.* A strongly positive (q,q)-form v is a convex combination of forms of type

$$(i\alpha_1 \wedge \bar{\alpha_1}) \wedge \cdots \wedge (i\alpha_q \wedge \bar{\alpha_q})$$

with $\alpha_j \in V^*$, for $j = 1, \cdots, q$. Of course, we are interested in the case $V = T_x X$. In this way, we are able to define, at each $x \in X$, a notion of positivity for smooth forms on X, and so we can then give a notion of positivity for currents. A current T of bidimension (p,p) is *positive* if $\langle T, u\rangle \geq 0$ for all strongly positive test forms $u \in \mathcal{D}^{p,p}(X)$.

Two extreme examples of closed positive currents are currents of integration along analytic subsets of dimension p and positive smooth closed differential forms of bidegree $(n-p, n-p)$.

Let T be a positive closed current of bidegree $(1,1)$ on a compact manifold X. Then T is locally given as $T = dd^c\varphi$ where φ is a plurisubharmonic (psh for short) function. This cannot hold globally since the maximum principle insures that the only psh functions on X are the constants. On the other hand, given θ a smooth representative of $\{T\}$, one can ask whether $T - \theta$ (which is d-exact) is also dd^c-exact. This is true in the Kähler setting:

Lemma 1.1.1 ($\partial\bar{\partial}$-Lemma)**.** *Let X be a compact Kähler manifold. Let S be a current which is both ∂ and $\bar{\partial}$-closed. Then S is d-exact if and only if is dd^c-exact.*

We refer the reader to [Voi07, Proposition 6.17] for a proof of the $\partial\bar{\partial}$-Lemma and to [Dem09] for more details about the notion of positive currents.

Consider now $\alpha \in H^{1,1}(X,\mathbb{R})$ a cohomology class which can be represented by a positive closed current (such a class is called *pseudoeffective*). Fix θ a smooth representative of α. If X is Kähler, then any closed positive current of bidegree $(1,1)$ in α can be written as

$$T = \theta + dd^c\varphi$$

for some upper semi-continuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$, which is uniquely determined up to an additive constant. Such functions are called θ-plurisubharmonic.

1.1.2 Quasi-plurisubharmonic functions

In this section we introduce the first basic properties of quasi-plurisubharmonic functions ([GZ05]). These are functions which are locally given as the sum of a smooth and a plurisubharmonic function. It follows from the maximum principle that on a compact manifold X, there are no global plurisubharmonic functions but the constants. However there is plenty of quasi-plurisubharmonic (qpsh for short) functions.

When (X,ω) is compact Kähler, any qpsh function φ is $A\omega$-psh for some $A > 0$ large enough, i.e. $A\omega + dd^c\varphi \geq 0$ in the weak sense of currents. Indeed, $dd^c\varphi$ is bounded from below by a smooth form, which is itself bounded from below by $-A\omega$, $A > 0$ large enough. By rescaling, one can assume $A = 1$. For this reason, here we restrict to consider ω-plurisubharmonic functions.

Definition 1.1.2. We let $\mathrm{PSH}(X,\omega)$ denote the set of ω-plurisubharmonic functions, i.e. the set of functions $\varphi \in L^1(X,\mathbb{R}\cup\{-\infty\})$ which write locally as the sum of a smooth and a plurisubharmonic function, and such that

$$\omega + dd^c\varphi \geq 0$$

in the weak sense of positive currents.

If $u : X \to \mathbb{R}$ is a function of class C^2, then $dd^c u$ is bounded from below by $-A\omega$ for some $A > 0$ large enough (which depends linearly on $\|u\|_{C^2}$). Up to rescaling, this shows that any smooth function is ω-plurisubharmonic. The set $\mathrm{PSH}(X, \omega)$ also contains singular functions. The local model of singular behavior is that of plurisubharmonic functions, but the ω-psh condition also encodes global information which limitates the possible type of singularities.

Quasi-plurisubharmonic functions have interesting compactness properties that are straighforward consequences of the analogous local results for sequences of psh functions.

Proposition 1.1.3. *Let* $(\varphi_j) \in \mathrm{PSH}(X, \omega)^{\mathbb{N}}$.

1) If (φ_j) *is uniformly bounded from above on* X*, then either* φ_j *converges uniformly to* $-\infty$ *on* X*, or the sequence* (φ_j) *is relatively compact in* $L^1(X)$.

2) If $\varphi_j \to \varphi$ *in* $L^1(X)$*, then* φ *coincides almost everywhere with a unique function* $\varphi^\star \in \mathrm{PSH}(X, \omega)$*. Moreover*

$$\limsup_{j \to +\infty} \varphi_j(x) \leq \varphi^\star(x),$$

with equality holding outside a pluripolar set, and

$$\lim_{j \to +\infty} \sup_X \varphi_j = \sup_X \varphi^\star.$$

3) In particular if φ_j *is decreasing, then either* $\varphi_j \to -\infty$ *or* $\varphi = \lim \varphi_j \in \mathrm{PSH}(X; \omega)$*. Similarly, if* φ_j *is increasing and uniformly bounded from above then* $\varphi := (\lim \varphi_j)^\star \in \mathrm{PSH}(X, \omega)$*, where* * *denotes the uppersemicontinuous regularization.*

4) If $\varphi_j \to \varphi$ *in the weak sense of distributions, then* φ *coincides almost everywhere with a unique function* $\varphi^\star \in \mathrm{PSH}(X, \omega)$ *and* $\varphi_j \to \varphi^\star$ *in* $L^p(X)$ *for any* $p > 1$*. Moreover the sequence* $\phi_j := (\sup_{l \geq j} \varphi_l)^\star$ *decreases to* $\varphi^\star$.

We refer the reader to [Dem09], chapter 1, for a proof. Note that 2) is a special case of Hartogs' lemma.

It is easy to approximate a given ω-psh function φ by a decreasing sequence of less singular ω-psh functions. One can for example consider $\varphi_p := -(-\varphi)^p$, where $0 < p < 1$ and φ is assumed to be normalized so that $\varphi \leq -1$. Letting p increase to 1 yields a decreasing family of ω-psh functions which are less singular. One can also approximate φ by a sequence of bounded ω-psh functions $\varphi_j := \max(\varphi; -j)$. It is more delicate to find a decreasing sequence of smooth ω-psh approximants ([Dem92, BK07]):

Proposition 1.1.4. *Fix* $\varphi \in \mathrm{PSH}(X, \omega)$*. Then there exists smooth* ω*-psh functions* $\varphi_j \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)$ *which decrease towards* φ.

We recall that quasi-psh functions are in $L^p(X)$ for every $p \geq 1$. A much more powerful integrability result, due to Skoda [Sko72], actually holds.

Theorem 1.1.5. *Fix* $\varphi \in \mathrm{PSH}(X,\omega)$ *and* $A < 2[\sup_{x\in X} \nu(\varphi,x)]^{-1}$. *Then* $\exp(-A\varphi) \in L^1(X)$. *Moreover if* $A < 2\nu(\{\omega\})$, *then*

$$sup \left\{ \int_X e^{-A\varphi} dV \mid \varphi \in \mathrm{PSH}(X,\omega) \text{ and } \sup_X \varphi = 0 \right\} < +\infty$$

We recall that the *Lelong number* of φ at a given point $x \in X$ is

$$\nu(\varphi,x) := \liminf_{z\to x} \frac{\varphi(x)}{\log \|z-x\|}$$

and

$$\nu(\{\omega\}) := \sup\{\nu(\varphi,x) \mid x \in X \text{ and } \varphi \in \mathrm{PSH}(X,\omega)\}.$$

One can check that $\nu(\{\omega\})$ is finite and only depends on the cohomology class of ω.

1.1.3 Positivity of cohomology classes

Let (X,ω) be a compact Kähler manifold. We introduce different notions of positivity for cohomology classes corresponding to convex cones in $H^{1,1}(X,\mathbb{R})$.

Definition 1.1.6. Let $\alpha \in H^{1,1}(X,\mathbb{R})$. Then

(i) α is a *pseudoeffective* class if and only if there exists a positive closed $(1,1)$-current T representing α.

(ii) α is a *nef* class (numerically effective) if and only if for all $\varepsilon > 0$, there exists a smooth and closed $(1,1)$-form $\theta_\varepsilon \in \alpha$ sucht that $\theta_\varepsilon \geq -\varepsilon\omega$.

(iii) α is a *big* class if and only if it can be represented by a Kähler current, i.e. a closed $(1,1)$-current T such that $T \geq \varepsilon\omega$ for $\varepsilon > 0$ small.

(iv) α is a *Kähler* class if and only if can be represented by a Kähler form, i.e. a smooth and closed $(1,1)$-form which is positive definite.

Clearly, such notions do not depend on the choice of the Kähler form ω since two Kähler forms are comparable.

The set of Kähler classes is an open convex cone $\mathcal{K} \subset H^{1,1}(X,\mathbb{R})$, the *Kähler cone*. Analogously, we can consider the *nef cone* $\mathcal{N}$ which is convex and closed, $\mathcal{B}$ the *big cone* which is convex and open and the *pseudoeffective cone* $\mathcal{E}$, convex and closed. Furthermore, the following inclusions hold:

$$\mathcal{K} \subset \mathcal{N} \subset \mathcal{E} \quad \text{and} \quad \mathcal{K} \subset \mathcal{B} \subset \mathcal{E},$$

with $\mathcal{K} = \mathring{\mathcal{N}}$ and $\mathcal{B} = \mathring{\mathcal{E}}$, where $^\circ$ denotes the interior.

Example 1.1.7. Let $\pi : X \to \mathbb{P}^2$ be the blow-up at one point p and $E = \pi^{-1}(p)$ be the exceptional divisor. Then it is known that

$$H^*(X) = \pi^\star H^*(\mathbb{P}^2) \oplus H^*(E).$$

In particular, this means that $H^{1,1}(X, \mathbb{R})$ is generated by $a\{\pi^\star \omega_{FS}\} + b\{E\}$ where $a, b \in \mathbb{R}$. The Kähler cone of X consists of all real $(1,1)$-cohomology classes α which are numerically positive on analytic cycles, i.e. such that $\int_Y \alpha^p > 0$ for every irreducible p-dimensional analytic set Y in X, ([DP04, Corollary 0.2]). It suffices to test this criterion with $Y = E$ and $Y = \mathbb{P}^1 \subset X$ where $\mathbb{P}^1$ intersects at most once the exceptional divisor E. We then get that

$$\mathcal{K} = \{a > 0, b < 0 \,|\, a > -b\}$$

and thus

$$\mathcal{N} = \{a \geq 0, b \leq 0 \,|\, a \geq -b\}.$$

Simply using the definition of the big cone we obtain

$$\mathcal{B} = \{a > 0, b \in \mathbb{R} \,|\, a > -b\}$$

and therefore

$$\mathcal{E} = \{a \geq 0, b \in \mathbb{R} \,|\, a \geq -b\}.$$

A compact complex manifold is Kähler iff its Kähler cone $\mathcal{K}$ is not empty. Given (Y, ω) a compact Kähler manifold and a modification $f : X \to Y$ with smooth center, although $f^\star \omega$ is not a Kähler form, X is a Kähler manifold:

Lemma 1.1.8. *Let $\pi : X \to Y$ be the blow up of Y with smooth connected center $\mathcal{Z}$ and E be the exceptional divisor. Assume ω is a Kähler form on Y. Then $\{\pi^\star \omega\} - \varepsilon\{E\}$ is a Kähler class on X, for every $0 < \varepsilon < 1$.*

We refer the reader to [Bla56, Theorem II.6].

1.1.4 Push-forward and Pull back

Let $f : X \to Y$ be a holomorphic map between two compact Kähler manifolds. One can push-forward a current S on X by duality (since the pull-back of a smooth differential form on Y is well defined), setting

$$\langle f_\star S, \eta \rangle := \langle S, f^\star \eta \rangle.$$

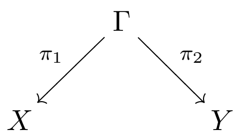
Observe that the push-forward preserves positivity, closedness and bidegree.

In general, given a current T on Y, it is not possible to define its pull-back by a holomorphic map. On the other hand, it is possible to pull-back positive closed currents of bidegree $(1,1)$. Indeed, such a current writes as $T = \theta + dd^c\varphi$, where $\theta \in \{T\}$ is a smooth form, and thus one can set

$$f^\star T := f^\star\theta + dd^c\varphi \circ f.$$

Clearly $f^\star T$ is a globally well defined current of bidegree $(1,1)$ on X which is closed and positive.

Moreover, we can define as well the push-forward and the pull-back of positive closed $(1,1)$-currents by a bimeromorphic map since any bimeromorphic map $f : X - - > Y$ can be decomposed as



where π_1, π_2 are two holomorphic and bimeromorphic maps and Γ denotes a desingularization of the graph of f.

Proposition 1.1.9. *Let X and Y be compact Kähler manifolds and let $\pi : X \to Y$ be the blow up of Y with smooth connected center $\mathcal{Z}$. Then*

1) given any positive closed $(1,1)$-current S on X, there exists a positive closed $(1,1)$-current T on Y such that

$$S = \pi^\star T + \gamma[E]$$

where E is the exceptional divisor and $\gamma \geq -\nu(T,\mathcal{Z})$, $\nu(T,\mathcal{Z}) := \inf_{x\in\mathcal{Z}} \nu(T,x)$. In particular given $\alpha_X \in H^{1,1}(X,\mathbb{R})$, we have the following decomposition at the level of cohomology classes

$$\alpha_X = \pi^\star\pi_\star\alpha_X + \gamma\{E\}$$

2) given any closed $(1,1)$-current T on Y and $\gamma \in \mathbb{R}$, the $(1,1)$-current $S : \pi^\star T + \gamma[E]$ is positive if and only if $\gamma \geq -\nu(T,\mathcal{Z})$.

Observe that the coefficient γ does not depend on the positive current S but only on its cohomology class.

We refer to [Dem09, Proposition 8.16, Corollary 2.14] and [Bou02a, Corollary 1.1.8] for a proof.

We are also interested in pushing-forward measures by holomorphic surjective maps. Let μ be a probability measure on X, then

$$f_\star\mu := \int_X \delta_{f(a)} d\mu(a).$$

In other words $f_\star\mu(E) = \mu(f^{-1}(E))$ for every Borel subset $E \subset Y$. The measure $f_\star\mu$ is a well defined probability measure.

We will say that a positive measure is *non-pluripolar* if it puts not mass on pluripolar sets. It is easy to check that such a property is preserved under push-forward.

1.2 Finite energy currents

1.2.1 Volume of big classes

Fix $\alpha \in H^{1,1}_{big}(X,\mathbb{R})$ and θ a smooth representative in α. Consider the θ-psh function

$$V_\theta := \sup\{\varphi \mid \varphi \in \mathrm{PSH}(X,\theta) \;\; \varphi \le 0 \;\text{ on } X\}.$$

Observe that if α is a Kähler or a semipositive class, then V_θ is bounded but this is not the case in general, if the class is merely big. On the other hand, by Demailly's regularization theorem [Dem92] it follows that there exists a Zariski open set $\Omega \subset X$ on which V_θ is locally bounded.

Moreover, we say that a positive $(1,1)$-current $T_{\min} = \theta + dd^c\varphi_{\min}$ in α has *minimal singularities* if $|\varphi_{\min} - V_\theta|$ is globally bounded.

We can then introduce

Definition 1.2.1. Let $T_{\min}$ be a current with minimal singularities in α. The positive number

$$\mathrm{vol}(\alpha) := \int_\Omega T^n_{min} \tag{1.2.1}$$

is called the *volume* of α.

Note that the Monge-Ampère measure of $T_{\min}$, i.e. the top wedge product of $T_{\min}$ is well defined in Ω thanks to Bedford and Taylor [BT87] and that the volume defined in (1.2.1) is independent of the choice of $T_{\min} \in \alpha$ (see [BEGZ10, Theorem 1.16]) and of the choice of Ω.

Volumes are invariant under modification between two Kähler manifolds but are not preserved in the case of push-forwards.

We recall that if α is a nef cohomology class then $\mathrm{vol}(\alpha) = \alpha^n$. This is not true when α is big but not nef.

Example 1.2.2. Consider $\pi : X \to \mathbb{P}^2$ the blow-up at one point and take $\alpha = \pi^\star\{\omega_{FS}\} + \{E\}$. In this case, it turns out that a current of minimal singularities in α is of the type $T_{\min} = \pi^\star S_{\min} + [E]$ where $S_{\min}$ is with minimal singularities in $\{\omega_{FS}\}$. In particular, V_θ locally writes as the sum of a bounded potential and $\log|x|$ where we choose local coordinates on X such that $E = \{x = 0\}$. Then $\mathrm{vol}(\alpha) = 1$ whereas $\alpha^2 = 0$.

1.2.2 The non-pluripolar product

Following Bedford and Taylor (see [BT87]), in [BEGZ10] the authors have introduced the non-pluripolar products of globally defined currents which is always well-defined on a compact Kähler manifold.

Fix α a big cohomology class on X. Given T a closed positive $(1,1)$-current in α, we fix $\theta \in \alpha$ a smooth form and we write $T = \theta + dd^c\varphi$. Consider now the "canonical approximants"

$$\varphi_j := \max(\varphi, V_\theta - j).$$

Then the sequence of Borel measures

$$\mathbf{1}_{\{\varphi > V_\theta - j\} \cap \Omega} (\theta + dd^c \varphi_j)^n$$

is non-decreasing and converges towards the so called *non pluripolar product* $\langle (\theta + dd^c\varphi)^n \rangle$.

Since by construction the non-pluripolar product does not put mass on pluripolar sets (and in particular on analytic sets), we have

$$\mathrm{vol}(\alpha) = \int_X \langle T^n_{\min} \rangle. \tag{1.2.2}$$

Observe that the total mass of the non-pluripolar measure $\langle T^n \rangle$ is less or equal to $\mathrm{vol}(\alpha)$.

Definition 1.2.3. We say that T has *full Monge-Ampère mass* if

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha)$$

and we denote by $\mathcal{E}(X, \alpha)$ the set of positive currents in α with full Monge-Ampère mass. We let $\mathcal{E}(X, \theta)$ denote the set of θ-psh functions such that $T = \theta + dd^c\varphi \in \mathcal{E}(X, \alpha)$.

Such currents have mild singularities in the *ample locus* $\mathrm{Amp}\,(\alpha)$, in particular they have zero Lelong number at every point $x \in \mathrm{Amp}\,(\alpha)$.
We recall that the ample locus of α is the set of points $x \in X$ such that there exists a strictly positive current $T \in \alpha$ with analytic singularities and smooth around x.

Similarly, we define *weighted energy classes*

$$\mathcal{E}_\chi(X, \theta) := \{\varphi \in \mathcal{E}(X, \theta) \,|\, \chi \circ \varphi \in L^1 (\langle (\theta + dd^c\varphi)^n \rangle)\}$$

where χ is a weight function. Here, by a *weight function*, we mean a smooth increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(-\infty) = -\infty$ and $\chi(t) = t$ for $t \geq 0$. We denote by $\mathcal{E}_\chi(X, \alpha)$ the set of positive currents whose potentials belong to $\mathcal{E}_\chi(X, \theta)$.

Example 1.2.4. Assume $X = \mathbb{P}^1$ and $\alpha = \{\omega_{FS}\}$ is normalized such that $\mathrm{vol}(\alpha) = 1$. A positive closed $(1,1)$-current $T \in \alpha$ is a probability measure that can be decomposed as $T = T_{polar} + T_{diffuse}$ where T_{polar} is the polar part of the measure (see [R$^+$69]). In this case $T \in \mathcal{E}(X, \alpha)$ is and only if $T_{polar} = 0$.

1.2.3 Stability of energy classes

We briefly describe in this section the contents of chapter 2. We study invariance properties of the energy classes $\mathcal{E}$ and $\mathcal{E}_\chi$.

Consider $f : X \to Y$ an holomorphic map between two Kähler manifolds and a big cohomology class β on Y, then it essentially follows from a change of coordinates that $\mathcal{E}(X, f^\star\beta) = f^\star(\mathcal{E}(Y,\beta))$ and $\mathcal{E}_\chi(X, f^\star\beta) = f^\star(\mathcal{E}_\chi(Y,\beta))$, for any weight function χ. One can then wonder what happens if we fix a big cohomology class α on X and we look at the push-forward of positive closed $(1,1)$-currents in α. More precisely we wonder wether

$$f_\star(\mathcal{E}(X,\alpha)) = \mathcal{E}(Y, f_\star\alpha)$$

and

$$f_\star(\mathcal{E}_\chi(X,\alpha)) = \mathcal{E}_\chi(Y, f_\star\alpha).$$

As we explain in what follows, things are more complicated in this case and we actually get the same type of results for any $f : X - - > Y$ merely bimeromorphic.

We stress that Kähler classes are not stable under bimeromorphic maps. The good objects to work with are big cohomology classes. Indeed, if α is big, then so are $f^\star\alpha$ and $f_\star\alpha$.

Theorem 1.2.5 (DN13)**.** *The non-pluripolar product is a bimeromorphic invariant. Given $f : X - - > Y$ a bimeromorphic map and α a big class, we have*

$$f_\star\langle T^n\rangle = \langle (f_\star T)^n\rangle$$

where $T \in \alpha$ is a positive $(1,1)$-current.

In general, finite energy classes are not preserved by bimeromorphic maps. For example, let $\pi : X \to \mathbb{P}^2$ be the blow-up at one point $\{p\}$ and $E = \pi^{-1}(p)$ be the exceptional divisor. On $\mathbb{P}^2$, we consider the Fubini-Study form ω_{FS} and a positive $(1,1)$-current ω' such that locally writes $dd^c \log\|z\|$. Then $\tilde{\omega} = (\pi^\star\omega' - [E]) + \pi^\star\omega_{FS}$ is a Kähler form and $\tilde{\omega} \in 2\pi^\star\omega_{FS} - \{E\} := \alpha$. In this case

$$\pi_\star(\mathcal{E}(X,\alpha)) \neq \mathcal{E}(\mathbb{P}^2, \pi_\star\alpha)$$

since $\tilde{\omega} \in \mathcal{E}(X,\alpha)$, but $\pi_\star\tilde{\omega} = \omega' + \omega_{FS}$ has positive Lelong number at p, and thus does not belong to $\mathcal{E}(\mathbb{P}^2, \pi_\star\alpha)$.

We introduce a natural condition to overcome this problem. We say that a big class α on X satisfies Condition (`V`) if

$$f_\star(\{\text{positive}\,(1,1)\text{-currents in}\,\alpha\}) = \{\text{positive}\,(1,1)\text{-currents in}\,f_\star\alpha\}.$$

Theorem 1.2.6 (DN13a)**.** *Let $\alpha \in H^{1,1}_{big}(X,\mathbb{R})$. If Condition (`V`) holds, then*

(i) $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha)$,

(ii) $f_\star(\mathcal{E}(X,\alpha)) = \mathcal{E}(Y, f_\star\alpha)$,

(iii) $f_\star(\mathcal{E}_\chi(X,\alpha)) = \mathcal{E}_\chi(Y, f_\star\alpha)$, *for any weight* χ.

Observe that in the previous example Condition (V) does not hold since $\text{vol}(\alpha) = 3 < \text{vol}(\pi_\star\alpha) = 4$.

In dimension 2, Condition (V) turns out to be equivalent to the preservation of volumes. We refer the reader to Section 2.2.2 for more details.

A related question is the stability of energy classes if we change cohomology classes on a fixed compact Kähler manifold X. More precisely, given α and β big classes, we ask wether

$$T \in \mathcal{E}_\chi(X,\alpha) \quad \textit{and} \quad S \in \mathcal{E}_\chi(X,\beta) \quad \overset{\Longrightarrow}{\Longleftarrow} \quad T + S \in \mathcal{E}_\chi(X, \alpha+\beta)$$

and similarly for weighted energy classes $\mathcal{E}_\chi$.

We have observed the following:

Theorem 1.2.7 (DN13b)**.** *Let* χ *be a weight function.*

(i) If α, β *are Kähler classes, then*

$$T \in \mathcal{E}_\chi(X,\alpha) \textit{ and } S \in \mathcal{E}_\chi(X,\beta) \textit{ if and only if } T+S \in \mathcal{E}_\chi(X,\alpha+\beta).$$

(ii) If α, β *are merely big classes, then*

$$T+S \in \mathcal{E}_\chi(X,\alpha+\beta) \textit{ implies } T \in \mathcal{E}_\chi(X,\alpha) \textit{ and } S \in \mathcal{E}_\chi(X,\beta).$$

The same statements hold for the energy class $\mathcal{E}$.

Furthermore, we show that the reverse implication in *(ii)* is false in general (see Counterexample 2.3.5).

1.2.4 Finite energy measures

We briefly describe the contents of Chapter 3. There we study the stability of finite energy measures.

Definition 1.2.8. We say that a probability measure μ has *finite energy* in a given class α (normalized such that $\text{vol}(\alpha) = 1$) if there exists $T \in \mathcal{E}^1(X,\alpha)$ such that

$$\mu = \langle T^n \rangle$$

and we write $\mu \in \text{MA}\,(\mathcal{E}^1(X,\alpha))$.

Example 1.2.9. When (X,ω) is a compact Riemann surface (i.e, $n=1$), it turns out that a probability measure $\mu = \omega + dd^c\varphi$ has finite energy if and only if $\nabla\varphi \in L^2(X)$.

This notion was introduced in [BBGZ13] where the authors defined the *electrostatic energy* of a probability measure $E^*(\mu)$. An equivalent formulation of having finite measure is that $E^*(\mu) < +\infty$.

Such pluricomplex energy is a natural analogue of the classical *logarithmic energy* of a measure in dimension 1. Recall that, given a probability measure on $\mathbb{C}$, its logarithmic energy $I(\mu)$ is defined by

$$I(\mu) := \int\int \log|z-w| d\mu(z) d\mu(w) = -\int p_\mu(z) d\mu(z)$$

where the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ is the *logarithmic potential* defined by

$$p_\mu := \int \log|z-w| d\mu(w).$$

Note that when μ has finite energy, p_μ belongs to $L^1(d\mu)$.

When $X = \mathbb{P}^1$ and ω_{FS} is the Fubini-Study form, it turns out that a given probability measure μ on $\mathbb{C} \subset \mathbb{P}^1$ is such that $E^*(\mu) < +\infty$ if and only if μ has finite logarithmic energy and in that case we have

$$E^*(\mu) = \frac{1}{2} I(\mu - \omega_{FS}).$$

We show that the notion of having finite measure is invariant by biholomorphisms but not by bimeromorphic maps.

Proposition 1.2.10 (DN14). *Let α, β be Kähler classes. Then*

$$\mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\alpha)) \Longleftrightarrow \mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\beta)).$$

The previous statement is false for big classes. Consider $\pi : X \to \mathbb{P}^2$ the blow-up at one point. We show that there exists a probability measure μ and a Kähler class $\{\tilde{\omega}\}$ on X such that

$$\mu \in \mathrm{MA}\,\left(\mathcal{E}^1(X, \{\tilde{\omega}\})\right) \quad \text{and} \quad \mu \notin \mathrm{MA}\,\left(\mathcal{E}^1(X, \{\pi^\star \omega_{FS}\})\right).$$

1.3 Kähler-Einstein metrics

1.3.1 The Calabi conjecture

Let X be an n-dimensional compact Kähler manifold and fix ω an arbitrary Kähler form. If we write locally

$$\omega = \frac{i}{\pi} \sum \omega_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta,$$

then the Ricci form of ω is (locally)

$$\mathrm{Ric}(\omega) := -\frac{i}{\pi} \sum \frac{\partial^2 \log(\det \omega_{pq})}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta.$$

Observe that $\mathrm{Ric}(\omega)$ is a closed $(1,1)$-form on X such that for any other Kähler form ω' on X, the following holds globally:

$$\mathrm{Ric}(\omega') = \mathrm{Ric}(\omega) - dd^c \log \frac{\omega'^n}{\omega^n}.$$

Here $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \bar{\partial})$ are both real operators. In particular $\mathrm{Ric}(\omega')$ and $\mathrm{Ric}(\omega)$ represent the same cohomology class, which turns out to be $c_1(X)$. Conversely, given η a closed differential form representing $c_1(X)$, Calabi asked in [Cal57] whether one can find a Kähler form ω such that

$$\mathrm{Ric}(\omega) = \eta.$$

He showed that if the answer is positive, then the solution is unique and proposed a continuity method to prove the existence. This problem, known as the Calabi conjecture, remained open for two decades. This result was finally solved by Yau in [Yau78] and is now known as the Calabi-Yau theorem.

The Calabi conjecture reduces to solving a complex Monge-Ampère equation as we can see here below. Fix $\alpha \in H^{1,1}(X,\mathbb{R})$ a Kähler class, ω a Kähler form in α and $\eta \in c_1(X)$ a smooth form. Since $\mathrm{Ric}(\omega)$ also represents $c_1(X)$, it follows from the $\partial\bar{\partial}$-lemma that there exists $h \in C^\infty(X,\mathbb{R})$ such that

$$\mathrm{Ric}(\omega) = \eta + dd^c h.$$

We now seek for $\omega_\varphi := \omega + dd^c\varphi$ a new Kähler form in α such that $\mathrm{Ric}(\omega_\varphi) = \eta$. Since

$$\mathrm{Ric}(\omega_\varphi) = Ric(\omega) - dd^c \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right),$$

the equation $\mathrm{Ric}(\omega_\varphi) = \eta$ is equivalent to

$$dd^c \left\{ h - \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right)\right\} = 0$$

The function inside the brackets is pluriharmonic, hence constant since X is compact. Shifting initially h by a constant, our problem is equivalent to solving the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^h \omega^n. \tag{CY}$$

Note that h necessarily satisfies the normalizing condition

$$\int_X e^h \omega^n = \int_X \omega^n = V.$$

Theorem 1.3.1 (Yau78)**.** *The equation (CY) admits a unique (up to constant) solution $\varphi \in C^\infty(X,\mathbb{R})$ such that ω_φ is a Kähler form.*

1.3.2 The Kähler-Einstein equation

The following metrics are objects of a great interest:

Definition 1.3.2. A Kähler metric ω is *Kähler-Einstein* if there exists $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(\omega) = \lambda\omega.$$

The existence of Kähler-Einstein metrics is one of the fundamental problems in complex geometry. It is easy to see that there exists obstructions to the existence of such metrics. Indeed, since $\{\mathrm{Ric}(\omega)\} = c_1(X)$, looking for a Kähler metric ω such that $\mathrm{Ric}(\omega) = \lambda\omega$ requires $c_1(X)$ to have a definite sign (the one of λ). This is always the case in dimension $n = 1$, but not necessarily so in dimension $n \geq 2$. For example, when $X = S_1 \times S_2$ is the product of two compact Riemann surfaces, then $c_1(X)$ is proportional to a Kähler class iff S_1 and S_2 are of the same type.

We recall that if $c_1(X) = c_1(K_X^{-1}) = -c_1(K_X)$ has a sign (non zero) then Kodaira's embedding theorem insures that the compact Kähler manifold X is actually projective.

Note that $\mathrm{Ric}(\varepsilon\omega) = \mathrm{Ric}(\omega)$ for any $\varepsilon > 0$, and hence there are essentially three cases to be considered, $\lambda \in \{-1, 0, 1\}$.

Fix $\lambda \in \mathbb{R}$ such that $\lambda\{\omega\} = c_1(X)$ and $h \in C^\infty(X, \mathbb{R})$ such that

$$\mathrm{Ric}(\omega) = \lambda\omega + dd^c h.$$

We now seek for a Kähler form $\omega_\varphi = \omega + dd^c\varphi$ such that $\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi$. Arguing as before we can reduce our problem to solving the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi + h}\omega^n. \tag{MA$_\lambda$}$$

One can always solve (MA$_\lambda$) when $\lambda < 0$, (X is then of general type), (Aubin-Yau theorem [Aub76, Yau78]) and when $\lambda = 0$ (the Calabi-Yau theorem [Yau78]). The solution is moreover (essentially) unique.

The situation is much more complicated when $\lambda > 0$: this is the case when X is a Fano manifold.

A compact complex manifold is called *Fano* if the anticanonical bundle K_X^{-1} is ample. Note that these are necessarily projective algebraic.

The only Fano Riemann surface is the Riemann sphere $\mathbb{P}^1$. If X is a 2-dimensional Fano manifold, then it is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at r points in general position, $0 \leq r \leq 8$. Fano manifolds of dimension two are called *DelPezzo* surfaces. Fano manifolds have also been classified in dimension 3 and there are 105 families. There are finitely many families in any dimension, but this number is becoming very large with n.

When X is Fano ($\lambda > 0$) there are obstructions to the existence of Kähler-Einstein metrics (e.g. the Futaki invariant has to vanish identically), and the solutions, when they exist, are moreover not unique (Bando-Mabuchi's theorem [BM87]).

Theorem 1.3.3 (BM87). *Let X be a Fano manifold and assume ω_0, ω_1 are Kähler-Einstein metrics. Then there exists $V \in H^0(X, TX)$, a holomorphic vector field whose flow ϕ_t connects ω_0 to $\omega_1 = \phi_1 \omega_0$.*

An important work of Tian [Tia90] settles the situation in dimension 2:

Theorem 1.3.4 (Tian90). *A smooth DelPezzo surface admits a Kähler-Einstein metric unless it is biholomorphic to $\mathbb{P}^2$ blown-up at one or two points.*

The situation in higher dimension ($n \geq 3$) has been an open problem until recently. It was conjectured for some time that the non-vanishing of the Futaki invariant was the only obstruction to the non-existence of a Kähler-Einstein metric. Some counter-examples were however produced by Tian in the 90's. It has been conjectured by Yau that a Fano manifold admits a Kähler-Einstein metric if and only if it is "stable" in some algebro-geometric sense. The conjecture was later on refined by Tian and Donaldson who extended it to the context of constant scalar curvature metrics. This conjecture has been solved by Chen, Donaldson, Sun [CDS12a, CDS12b, CDS13] and Tian [Tia12] who proved the existence of Kähler-Einsten metrics on a Fano manifold if and only if the manifold is K-stable.

1.3.3 Continuity method

The continuity method is a classical tool to try and solve non linear PDE's. It consists in deforming the PDE of interest into a simpler one for which one already knows the existence of a solution. The following path of equations was proposed by Aubin

$$(\omega + dd^c \varphi_t)^n = e^{-\lambda \varphi_t + th} \omega^n \tag{MA$_t$}$$

where $0 \leq t \leq 1$ and $\varphi \in \mathrm{PSH}(X, \omega)$. The equation of interest corresponds to $t = 1$ while $(\mathrm{MA})_0$ admits the obvious solution $\varphi_0 = 0$.

The goal is then to show that the set $S \subset [0,1]$ of parameters for which there is a (smooth) solution is both open and closed in $[0,1]$ and since $[0,1]$ is connected and $0 \in S$, it will then follow that $S = [0,1]$ hence $1 \in S$.

The openness follows by linearizing the equation (this involves the Laplace operator associated to $\omega_\varphi = \omega + dd^c\varphi$) and using the implicit function theorem. Although it is not completely trivial this is not the most difficult part.

One then needs to establish various a priori estimates to show that S is closed. Indeed, if we consider a sequence of smooth solutions φ_t, as $t \to t_\infty \in (0,1]$, we want to extract a subsequence t_k such that φ_{t_k} converges uniformly with all its derivatives to a function φ_{t_∞} solution of $(\mathrm{MA})_\infty$. By Ascoli's thorem, it suffices to obtain *a priori* estimates of the type

$$\|\varphi_t\|_{C^k} \leq C_k$$

where C_k is a positive constant that depends only on k (it does not depend on t).

Observe that the previous arguments hold for all $\lambda \in \mathbb{R}$. It is at the level of C^0-estimates that the sign of λ plays a crucial role. In the case of negative curvature ($\lambda < 0$) a simple application of the maximum principle allows to conclude. When $\lambda = 0$, the situation is much more complicated: the C^0-estimates in this case are due to Yau and the approach relies on Moser's iterative process. After the celebrated paper of Yau [Yau78], Kołodziej [Koł98] generalized the C^0 a priori estimates using pluripotential tools. His uniform estimate can indeed be applied to complex Monge-Ampère equations of the type

$$(\omega + dd^c\varphi) = f dV$$

where $0 \leq f \in L^p(dV)$ for some $p > 1$.

Kołodziej's idea is to show that the Monge-Ampère capacity of sublevel sets $(\varphi < -t)$ vanishes if $t > 0$ is large enough, by a clever use of the comparison principle.

Using Kołodziej's method the regularity theory was also extended to the case when the reference cohomology class is non Kähler [EGZ09, BEGZ10].

Finally, in the case of positive curvature, such estimates do not exist in general and the continuity method stops at $t_\infty < 1$.

We assume from now on $\lambda \leq 0$. Once one has in hands C^0-estimates, one needs higher order estimates. The first step is to obtain a laplacian estimate, in other words we want to show

$$C^{-1}\omega \leq \omega + dd^c\varphi_t \leq C\omega$$

for some constant $C > 0$ that is independent of t. Then, thanks to Evans-Krylov theory, we can deduce an estimate of type $C^{2,\alpha}$ and this suffices to apply Schauder's theorems and a bootstrap argument.

1.4 More Monge-Ampère equations

We briefly describe in this section the contents of Chapters 4 and 5.

1.4.1 The quasi-projective setting

The second part of this thesis is devoted to study complex Monge-Ampère equations on complex quasi-projective varieties. More specifically, we consider

$D \subset X$ a divisor in our compact Kähler manifold and we look at complex Monge-Ampère equations of the type

$$(\omega + dd^c\varphi)^n = f\omega^n \tag{1.4.1}$$

where the density $0 < f \in C^\infty(X \setminus D)$.

Observe that if $f \in C^\infty(X)$ then the solution φ is also smooth on X, thanks to Yau's result.

One can try and study the regularity of the solution φ and its asymptotic behavior near D. We recall that the existence and the uniqueness of a weak solution of (1.4.1) follow from a general theory developed in the last years ([GZ07, Din09]).

The main result in the paper [DNL14a] in collaboration with Hoang Chinh Lu is the following:

Theorem 1.4.1 (DN-Lu14). *Assume $0 < f \in C^\infty(X \setminus D)$. If moreover $f = e^{\psi^+ - \psi^-}$, where $\psi^\pm$ are quasi-plurisubharmonic functions and $\psi^- \in L^\infty_{loc}(X \setminus D)$, then there exists a unique (up to an additive constant) solution φ of (1.4.1) which is smooth in $X \setminus D$.*

The strategy of the proof is to use Demailly's regularization theorem to approximate $\psi^\pm$ by smooth qpsh functions $\psi^\pm_\varepsilon$ on X. By Yau's theorem we know that there exists $\varphi_\varepsilon \in C^\infty(X)$ unique solution of

$$(\omega + dd^c\varphi_\varepsilon)^n = c_\varepsilon e^{\psi^+_\varepsilon - \psi^-_\varepsilon}\omega^n$$

with the normalization $\sup_X \varphi_\varepsilon = 0$. Here $c_\varepsilon > 0$ is a normalization constant such that the compability condition holds, i.e.

$$\int_X c_\varepsilon e^{\psi^+_\varepsilon - \psi^-_\varepsilon}\omega^n = \int_X \omega^n.$$

The first step is a uniform C^0-estimate (see Section 1.4.2 for a detailed explanation of this crucial step). Once we have it in hands we are able to get the laplacian estimate

$$\Delta_\omega \varphi_\varepsilon \le Ae^{-2\psi^-},$$

where A is a positive constant depending only on $\int_X e^{-C\varphi}\omega^n$, $C > 0$. Let us stress that, since $\varphi \in \mathcal{E}(X, \omega)$, such an integral is finite for any $C > 0$ thanks to Skoda's theorem.

As we have already explained, laplacian estimates and the ellipticity of Monge-Ampère operator allow to obtain higher order estimates for any compact subset $K \subset X \setminus D$,

$$\|\varphi_\varepsilon\|_{C^{k,\beta}}(K) \le C_{K,\alpha,\beta}$$

for any $k \ge 2$, $\beta \in (0, 1)$.

As in the "classical" case, the C^0-estimate is the most difficult one. We state here a general result that covers in particular the previous setting:

Theorem 1.4.2 (DN-Lu14). *Assume $f \lesssim e^{-\phi}$ where ϕ is a quasi-psh function. Let φ be the unique solution of (1.4.1) with $\sup_X \varphi \leq 0$. Then for any $a > 0$ such that $a\phi \in \mathrm{PSH}(X, \omega/2)$, there exists $A > 0$ depending only on $\int_X e^{-2\varphi/a}\omega^n$ such that*

$$\varphi \geq a\phi - A.$$

Observe that at this level, no regularity assumptions on the density f are required.

1.4.2 Monge-Ampère capacities

In order to give an idea of the proof of Theorem 1.4.2, we start explaining Kołodziej's techniques from pluripotential theory. The Monge-Ampère capacity of any Borel set $E \subset X$ is

$$\mathrm{Cap}_\omega(E) := \sup\left\{\int_E (\omega + dd^c u)^n \mid u \in \mathrm{PSH}(X,\omega) \ \ -1 \leq u \leq 0\right\}.$$

The capacity Cap_ω is comparable to the classical Monge-Ampère capacity of Bedford and Taylor and characterizes pluripolar sets. In Kołodziej's approach the idea is to prove that the function $H(t) := \mathrm{Cap}_\omega(\{\varphi < -t\})^{1/n}$ satisfies

$$sH(t+s) \leq CH(t)^2, \qquad \forall t > 0, s \in (0,1) \tag{1.4.2}$$

where $C > 0$ is a uniform constant. Such an inequality allows to deduce that there exists $t_\infty > 0$ such that

$$\mathrm{Cap}_\omega(\{\varphi < -t\}) = 0, \qquad \forall t \geq t_\infty$$

and therefore there exists $A > 0$ such that $\varphi \geq -A$. Hence φ is globally bounded.

In our case the solution is not bounded and therefore a natural idea is to bound the solution from below by a singular "model" qpsh function. This is the reason why in the works with Lu [DNL14a, DNL14b], we have introduced and studied *generalized Monge-Ampère capacities*, for example of type

$$\mathrm{Cap}_\psi(E) := \sup\left\{\int_E (\omega + dd^c u)^n \mid \psi - 1 \leq u \leq \psi\right\}, \quad \forall E \subset X$$

where ψ is a $\omega/2$-psh function that can be singular.

The goal is, once again, to prove an inequality as in (1.4.2) where now

$$H(t) := \mathrm{Cap}_\psi(\{\varphi < \psi - t\})^{1/n}.$$

1.4.3 Behavior near a divisor

It is natural for various geometric reasons to look at complex Monge-Ampère equations as in (1.4.1) where the density in the right side is smooth outside a divisor $D \subset X$.

As a particular case consider complete metrics in $X \setminus D$ of Poincaré type, namely when the density

$$f = \frac{h}{|s_D|^2(-\log|s_D|)^2}$$

with $h \in C^\infty(X \setminus D)$. In a recent work Auvray [Auv11] has proved an interesting result: he assumes h to have a very precise asymptotic behavior and regularity and shows that the solution φ looks like $-\log(-\log|s_D|)$. Our C^0-estimate works in this case just applying Theorem 1.4.2 with $\phi = 2\log|s_D|$ (actually we can get even better bounds of the solution, see Section 4.3.2 and Proposition 4.3.5).

We can treat as well densities f such that

$$f \leq \frac{B}{|s_D|^2(-\log|s_D|)^{1+\alpha}}$$

where $B, \alpha > 0$. As in the case of the Poincaré metric, in these special cases we are able to say more about the behavior of φ.

Kołodziej's result covers the case of such densities when $\alpha > n$ proving that the solution φ is globally bounded. With our method we can improve Kołodziej's result and show that φ is globally bounded when $\alpha > 1$.

Moreover, if $\alpha \leq 1$ we prove that the solution is not bounded and we are able to give a precise lower bound: if $\alpha \leq 1$, for each $q \in (1-\alpha, 1)$ we have $\varphi \geq -(-\log|s_D|)^q - C$; furthermore if $\alpha = 1$ and D is smooth, $\varphi \geq -\log(-\log|s_D|) - C$.

1.4.4 Non integrable densities

Berman and Guenancia [BG13] have studied the existence of singular Kähler-Einstein metrics on general type varieties with log-canonical singularities.

We recall that singular varieties with log-canonical singularities naturally appear in the compactification of moduli space of non-singular projective algebraic varieties with ample canonical bundle. Such a problem is related to the Minimal Model Program in birational geometry.

Berman and Guenancia's problem reduces to a complex Monge-Ampère equation of the type

$$\mathrm{MA}(\varphi) = e^\varphi f dV \tag{1.4.3}$$

where $f \sim \frac{1}{|s_D|^2}$ and s_D is a holomorphic section defining the divisor D.

In Chapter 5 we also look at Monge-Ampère equations of the type

$$(\omega + dd^c\varphi)^n = e^\varphi f \omega^n. \tag{1.4.4}$$

When $0 \leq f \notin L^1(X)$ it is not clear that we can find a solution $\varphi \in \mathcal{E}(X,\omega)$ of equation (1.4.4). Using our generalized Monge-Ampère capacities, we show that it suffices to find a subsolution. Furtheremore, in the same spirit of what we have done in Chapter 4, if the density f is smooth outside a divisor D, we provide the regularity of a solution (whenever it exists) outside D.

Theorem 1.4.3 (DN-Lu14). *Let $0 \leq f$ be a measurable function such that $\int_X f\omega^n = +\infty$. If there exists $u \in \mathcal{E}(X,\omega)$ such that $\mathrm{MA}\,(u) \geq e^u f\omega^n$ then there is a unique $\varphi \in \mathcal{E}(X,\omega)$ such that*

$$\mathrm{MA}\,(\varphi) = e^{\varphi} f\omega^n.$$

Moreover, if $0 < f \in C^\infty(X \setminus D)$ and $f = e^{\psi^+ - \psi^-}$, where $\psi^\pm$ are quasi-plurisubharmonic functions and $\psi^- \in L^\infty_{loc}(X \setminus D)$, then φ is smooth on $X \setminus D$.

Observe that if $f = \frac{1}{|s_D|^2}$ then there exists suitable positive constants C_1, C_2 such that the function $\varphi = -2\log(-\log|s_D| + C_1) - C_2$ is a subsolution of $\mathrm{MA}\,(\varphi) = \frac{e^{\varphi}}{|s_D|^2}\omega^n$ (see Examples 5.3.7 and 5.3.9). Our result thus covers such cases.

Chapter 2

Stability of Monge-Ampère energy classes

Introduction

Let X be a compact n-dimensional Kähler manifold, $T_1 = \theta_1 + dd^c\varphi_1, ..., T_p = \theta_p + dd^c\varphi_p$ be closed positive $(1,1)$-currents and $\theta_1 + dd^c V_{\theta_1}, ..., \theta_p + dd^c V_{\theta_p}$ be canonical currents with minimal singularities. Following the construction of Bedford-Taylor [BT87] in the local setting, it has been shown in [BEGZ10] that

$$\mathbf{1}_{\bigcap_j \{\varphi_j > V_{\theta_j} - k\}}(\theta_1 + dd^c \max(\varphi_1, V_{\theta_1} - k)) \wedge ... \wedge (\theta_p + dd^c \max(\varphi_p, V_{\theta_p} - k))$$

is non-decreasing in k and converge to the so called *non-pluripolar product*

$$\langle T_1 \wedge ... \wedge T_p \rangle.$$

The resulting positive (p,p)-current does not charge pluripolar sets and it is always *well-defined* and *closed*.

Given α a big cohomology class, a positive closed $(1,1)$-current $T \in \alpha$ is said to have *full Monge-Ampère mass* if

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha)$$

and we then write $T \in \mathcal{E}(X,\alpha)$. In [BEGZ10] the authors define also *weighted energy functionals* E_χ (for any weight χ) in the general context of a big class extending the case of a Kähler class ([GZ07]). The space of currents with finite weighted energy is denoted by $\mathcal{E}_\chi(X,\alpha)$.

The aim of the present paper is to show the invariance of the non-pluripolar product and establish stability properties of energy classes.

Theorem A. *The non-pluripolar product is a bimeromorphic invariant.* More precisely, fix $\alpha \in H^{1,1}(X,\mathbb{R})$ a big class and $f : X - - > Y$ a bimeromorphic map, then

1) $f_\star\langle T^n\rangle = \langle (f_\star T)^n\rangle$ for any positive closed $T \in \alpha$.

Furthermore if $f_\star\Big(\mathcal{T}_\alpha(X)\Big) = \mathcal{T}_{f_\star\alpha}(Y)$ then

2) $f_\star(\mathcal{E}(X,\alpha)) = \mathcal{E}(Y, f_\star\alpha)$;

3) $f_\star(\mathcal{E}_\chi(X,\alpha)) = \mathcal{E}_\chi(Y, f_\star\alpha)$ for any weigth $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$.

Here $\mathcal{T}_\alpha(X)$ denotes the set of all positive and closed currents in the big class α and $\mathcal{T}_{f_\star\alpha}(Y)$ is the set of all positive closed currents in the image class. The Condition on the image of positive currents insures that the push-forward of a current with minimal singularities is still with minimal singularities: this easily implies that the volumes are preserved, i.e. $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha)$. We show conversely in Propostion 2.2.5 that the condition $f_\star\Big(\mathcal{T}_\alpha(X)\Big) = \mathcal{T}_{f_\star\alpha}(Y)$ is equivalent to $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha)$ in complex dimension 2, by using the existence of Zariski decompositions.

A related problem is to understand what happens to the energy classes if we change cohomology classes on a fixed compact Kähler manifold. Let α, β be big cohomology classes. Given $T \in \mathcal{T}_\alpha(X)$ and $S \in \mathcal{T}_\beta(X)$ so that $T + S \in \mathcal{T}_{\alpha+\beta}(X)$, we wonder whether

$$T \in \mathcal{E}_\chi(X,\alpha) \quad \textit{and} \quad S \in \mathcal{E}_\chi(X,\beta) \quad \overset{\Longrightarrow}{\Longleftarrow} \quad T + S \in \mathcal{E}_\chi(X,\alpha+\beta)$$

It turns out that $T + S \in \mathcal{E}_\chi(X,\alpha+\beta)$ implies $T \in \mathcal{E}_\chi(X,\alpha)$ and $S \in \mathcal{E}_\chi(X,\beta)$ in a very general context (Proposition 2.3.1) but the reverse implication is false in general (see Counterexamples 2.3.5 and 2.3.7). We obtain a positive answer under restrictive conditions on the cohomology classes (see Propositions 2.3.3 and 2.4.8).

Theorem B. Let α, β be merely big classes, $T \in \mathcal{T}_\alpha(X)$, $S \in \mathcal{T}_\beta(X)$ and $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$. Then

1) $T + S \in \mathcal{E}(X,\alpha+\beta)$ implies $T \in \mathcal{E}(X,\alpha)$ and $S \in \mathcal{E}(X,\beta)$,

2) $T + S \in \mathcal{E}_\chi(X,\alpha+\beta)$ implies $T \in \mathcal{E}_\chi(X,\alpha)$ and $S \in \mathcal{E}_\chi(X,\beta)$.

If α, β are Kähler, conversely

3) $T \in \mathcal{E}(X,\alpha)$ and $S \in \mathcal{E}(X,\beta)$ implies $T + S \in \mathcal{E}(X,\alpha+\beta)$,

4) $T \in \mathcal{E}_\chi(X,\alpha)$ and $S \in \mathcal{E}_\chi(X,\beta)$ implies $T + S \in \mathcal{E}_\chi(X,\alpha+\beta)$.

Proposition C. Assume that $S \in \beta$ has bounded local potentials and that the sum of currents with minimal singularities in α and in β is still with minimal singularities. If $p > n^2 - 1$ then

$$T \in \mathcal{E}^p(X,\alpha) \Longrightarrow T + S \in \mathcal{E}^q(X,\alpha+\beta),$$

where $0 < q < p - n^2 + 1$.

We stress that the condition on the sum of currents having minimal singularities is not always satisfied as noticed in Remark 2.3.8, but it is a necessary condition if we want the positive intersection class $\langle \alpha \cdot \beta \rangle$ to be multi-linear (see [BEGZ10]). In our proof of Proposition **C** we establish a comparison result of capacities which is of independent interest:

Theorem D. Let α be a big class and β be a semipositive class. We assume that the sum of currents with minimal singularities in α and β is still with minimal singularities. Then, for any Borel set $K \subset X$, there exist $C > 0$ such that

$$\frac{1}{C} Cap_{\theta_{\alpha,\min}}(K) \leq Cap_{\theta_{\alpha+\beta,\min}}(K) \leq C \left(Cap_{\theta_{\alpha,\min}}(K) \right)^{\frac{1}{n}}$$

where $\theta_{\alpha,\min} := \theta_\alpha + dd^c V_{\theta_\alpha}$.

Let us now describe the contents of the article. We first introduce some basic notions such as currents with minimal singularities and finite energy classes and we recall more or less known facts, e.g. that currents with full Monge-Ampère mass have zero Lelong number on a Zariski open set (Proposition 2.1.9).

In Section 2.2, we show that the non-pluripolar product is a bimeromorphic invariant (Theorem 2.2.1). Furthermore, under a natural condition on the set of positive $(1,1)$-currents, we are able to prove that weighted energy classes are preserved under bimeromorphic maps (Proposition 2.2.3).

In the third part of the paper we study the stability of the energy classes (see e.g. Theorem 2.3.1 and Proposition 2.3.3) and we give some counterexamples.

Finally, we compare the Monge-Ampère capacities w.r.t different big classes (Theorem 2.4.6) and we use this result to give a partial positive answer to the stability property of weighted homogeneous classes $\mathcal{E}^p$ (Proposition 2.4.8).

2.1 Preliminaries

2.1.1 Big classes

Let X be a compact Kähler manifold and let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a real $(1,1)$-cohomology class.

Recall that α is said to be *pseudo-effective* (*psef* for short) if it can be represented by a closed positive $(1,1)$-current T. Given a smooth representative θ of the class α, it follows from $\partial\bar{\partial}$-lemma that any positive $(1,1)$-current can be written as $T = \theta + dd^c\varphi$ where the global potential φ is a θ-psh function, i.e. $\theta + dd^c\varphi \geq 0$. Here, d and d^c are real differential operators defined as

$$d := \partial + \bar{\partial}, \qquad d^c := \frac{i}{2\pi}\left(\bar{\partial} - \partial\right).$$

The set of all psef classes forms a closed convex cone and its interior is by definition the set of all *big* cohomology classes:

Definition 2.1.1. We say that α is *big* if it can be represented by a *Kähler current*, i.e. there exists a positive closed $(1,1)$-current $T \in \alpha$ that dominates a Kähler form .

Analytic and minimal singularities

A positive current $T = \theta + dd^c\varphi$ is said to have *analytic singularities* if there exists $c > 0$ such that (locally on X),

$$\varphi = \frac{c}{2} \log \sum_{j=1}^{N} |f_j|^2 + u,$$

where u is smooth and $f_1, ...f_N$ are local holomorphic functions.

Definition 2.1.2. *If α is a big class, we define its* ample locus $\mathrm{Amp}\,(\alpha)$ *as the set of points $x \in X$ such that there exists a strictly positive current $T \in \alpha$ with analytic singularities and smooth around x.*

The ample locus $\mathrm{Amp}\,(\alpha)$ is a Zariski open subset by definition, and it is nonempty thanks to Demaillly's regularization result (see [Bou04]).

If T and T' are two closed positive currents on X, then T is said to be *more singular* than T' if their local potentials satisfy $\varphi \leq \varphi' + O(1)$.

Definition 2.1.3. *A positive current T is said to have* minimal singularities *(inside its cohomology class α) if it is less singular than any other positive current in α. Its θ-psh potentials φ will correspondingly be said to have minimal singularities.*

Such θ-psh functions with minimal singularities always exist, one can consider for example

$$V_\theta := \sup \{\varphi \ \theta\text{-psh}, \varphi \leq 0 \text{ on } X\}.$$

Remark 2.1.4. Let us stress that the sum of currents with minimal singularities does not necessarily have minimal singularities. For example, consider $\pi : X \to \mathbb{P}^2$ the blow up at one point p and set $E := \pi^{-1}(p)$. Take $\alpha = \pi^\star\{\omega_{FS}\} + \{E\}$ and $\beta = 2\pi^\star\{\omega_{FS}\} - \{E\}$ where ω_{FS} denotes the *Fubini-Study* form on $\mathbb{P}^2$. As we will see in Remark 2.2.4 currents with minimal singularities in α are of the form $S_{\min} = \pi^\star T_{\min} + [E]$ where $T_{\min}$ is a current with minimal singularities in $\{\omega_{FS}\}$ (i.e. its potential is bounded) and so they have singularities along E. On the other hand, currents with minimal singularities in the Kähler class β have bounded potentials, hence the sum of currents with minimal singularities in α and in β is a current with unbounded potentials. But $\alpha + \beta = 3\pi^\star\{\omega_{FS}\}$ is semipositive hence currents with minimal singularities have bounded potentials.

Images of big classes.

It is classical that big cohomology classes are invariant under pull back and push forward (see e.g. [Bou02b, Proposition 4.13]).

Lemma 2.1.5. *Let $f : X - - > Y$ be a bimeromorphic map and $\alpha_X \in H^{1,1}(X,\mathbb{R})$, $\alpha_Y \in H^{1,1}(Y,\mathbb{R})$ be big cohomology classes. Then $f_\star\alpha_X$ and $f^\star\alpha_Y$ are still big classes.*

Note that this is not true in the case of Kähler classes.

Volume of big classes.

Fix $\alpha \in H^{1,1}_{big}(X,\mathbb{R})$. We introduce

Definition 2.1.6. *Let $T_{\min}$ a current with minimal singularities in α and let Ω a Zariski open set on which the potentials of $T_{\min}$ are locally bounded, then*

$$\mathrm{vol}(\alpha) := \int_\Omega T^n_{\min} > 0 \tag{2.1.1}$$

is called the volume of α.

Note that the Monge-Ampère measure of $T_{\min}$ is well defined in Ω by [BT82] and that the volume is independent of the choice of $T_{\min}$ and Ω ([BEGZ10, Theorem 1.16]).

Let $f : X \to Y$ be a modification between compact Kähler manifolds and let $\alpha_Y \in H^{1,1}(Y,\mathbb{R})$ be a big class. The volume is preserved by pull-backs,

$$\mathrm{vol}(f^\star\alpha_Y) = \mathrm{vol}(\alpha_Y)$$

(see [Bou02b]), on the other hand, it is in general not preserved by push-forwards:

Example 2.1.7. Let $\pi : X \to \mathbb{P}^2$ be the blow-up along $\mathbb{P}^2$ at point p. The class $\alpha_X := \{\pi^\star\omega_{FS}\} - \varepsilon\{E\}$ is Kähler whenever $0 < \varepsilon < 1$ and $\pi_\star\alpha_X = \{\omega_{FS}\}$. Now, $\mathrm{vol}(\alpha_X) = 1 - \varepsilon^2$ while $\mathrm{vol}(\pi_\star\alpha_X) = 1$.

2.1.2 Finite energy classes

Fix X a n-dimensional compact Kähler manifold, $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big class and $\theta \in \alpha$ a smooth representative.

The non-pluripolar product

Let us stress that since the non-pluripolar product does not charge pluripolar sets,

$$\mathrm{vol}(\alpha) = \int_X \langle T^n_{\min}\rangle.$$

Definition 2.1.8. *A closed positive* $(1,1)$*-current* T *on* X *with cohomology class* α *is said to have* full Monge-Ampère mass *if*

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha).$$

We denote by $\mathcal{E}(X,\alpha)$ *the set of such currents. If* φ *is a* θ*-psh function such that* $T = \theta + dd^c\varphi$*. The* non-pluripolar Monge-Ampère measure *of* φ *is*

$$\mathrm{MA}\,(\varphi) := \langle (\theta + dd^c\varphi)^n \rangle = \langle T^n \rangle.$$

We will say that φ *has* full Monge-Ampère mass *if* $\theta + dd^c\varphi$ *has full Monge-Ampère mass. We denote by* $\mathcal{E}(X,\theta)$ *the set of corresponding functions.*

Currents with full Monge-Ampère mass have mild singularities.

Proposition 2.1.9. *A closed positive* $(1,1)$*-current* $T \in \mathcal{E}(X,\alpha)$ *has zero Lelong number at every point* $x \in \mathrm{Amp}\,(\alpha)$*.*

Proof. This is an adaptation of [GZ07, Corollary 1.8]. Let us denote $\Omega = \mathrm{Amp}\,(\alpha)$. We claim that for any compact $K \subset\subset \Omega$ there exists a positive closed $(1,1)$-current $T_K \in \alpha$ with minimal singularities and such that it is a smooth Kähler form near K. Fix θ a smooth form in α and $T_{\min} = \theta + dd^c\varphi_{\min}$ a current with minimal singularities. By Demailly's regularization theorem [Dem92], in the big class α we can find a strictly positive current with analytic singularities $T_0 = \theta + dd^c\varphi_0$ that is smooth on Ω. Then we define

$$\varphi_C := \max(\varphi_0, \varphi_{\min} - C)$$

where $C >> 1$. Clearly, $T_C = \theta + dd^c\varphi_C$ is the current we were looking for. For any point $x \in \Omega$, let $K = \overline{B(x,r)}$. Let χ be a smooth cut-off function on X such that $\chi \equiv 1$ on $B(x,r) \subset K$ and $\chi \equiv 0$ on $X \setminus B(x,2r)$ where $r > 0$ is small. Consider a local coordinates system in a neighbourhood of x and define the θ-psh function $\psi_\varepsilon = \varepsilon\chi \log\|\cdot\| + \varphi_C$ for ε small enough. Now, if $T = \theta + dd^c\varphi$ has positive Lelong number at point x, then $\varphi \leq \psi_\varepsilon$. On the other hand $T_\varepsilon = \theta + dd^c\psi_\varepsilon$ does not have full Monge-Ampère mass since

$$\int_{\{\psi_\varepsilon \leq \varphi_C - k\} \cap B(x,r)} \mathrm{MA}\,(\psi_\varepsilon^{(k)})$$

does not converge to 0 as k goes to $+\infty$, where $\psi_\varepsilon^{(k)} := \max(\psi_\varepsilon, \varphi_C - k)$ are the "canonical" approximants of ψ_ε ([BEGZ10, p.229]). Therefore by [BEGZ10, Proposition 2.14], it follows that $T \notin \mathcal{E}(X,\alpha)$. □

We say that a positive closed $(1,1)$-current $T \in \alpha$ is pluripolar if it is supported by some closed pluripolar set: if $T = \theta + dd^c\varphi$, T is pluripolar implies that $\mathrm{supp}\, T \subset \{\varphi = -\infty\}$.

Lemma 2.1.10. *For $j = 1, ..., p$, let $\alpha_j \in H^{1,1}(X, \mathbb{R})$ be a big class and $T_j \in \alpha_j$. If T_1 is pluripolar then*

$$\langle T_1 \wedge ... \wedge T_p \rangle = 0.$$

Proof. First note that, since the non pluripolar product does not put mass on pluripolar sets, we have

$$\mathbf{1}_{X \setminus A} \langle T_1 \wedge ... \wedge T_n \rangle = \langle T_1 \wedge ... \wedge T_n \rangle$$

with A the closed pluripolar set supporting T_1. Now, let ω be a Kähler form on X. In view of [BEGZ10, Proposition 1.14], upon adding a large multiple of ω to the T_j's we may assume that their cohomology classes are Kähler classes. We can thus find Kähler forms ω_j such that $T_j = \omega_j + dd^c \varphi_j$. Let U be a small open subset of $X \setminus A$ on which $\omega_j = dd^c \psi_j$, where $\psi_j \leq 0$ is a smooth psh function on U, so that $T_j = dd^c u_j$ on U. By definition on the plurifine open subset

$$O_k := \bigcap_j \{u_j > -k\}$$

we must have $\mathbf{1}_{O_k} \langle dd^c u_1 \wedge ... \wedge dd^c u_p \rangle = \mathbf{1}_{O_k} \bigwedge_j dd^c \max(u_j, -k)$. Since u_1 is a smooth potential on U, $u_1 > -k$ for k big enough and furthermore, since T_1 is supported by A, we have that $dd^c u_1 = 0$. So, clearly

$$\mathbf{1}_{O_k} \bigwedge_j dd^c \max(u_j, -k) = 0$$

and hence the conclusion. □

Weighted energy classes

By a *weight function*, we mean a smooth increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi(0) = 0$ and $\chi(-\infty) = -\infty$. We let

$$\mathcal{W}^- := \left\{ \chi : \mathbb{R}^- \to \mathbb{R}^- \mid \chi \text{ convex increasing, } \chi(0) = 0, \chi(-\infty) = -\infty \right\}$$

and

$$\mathcal{W}^+ := \left\{ \chi : \mathbb{R}^- \to \mathbb{R}^- \mid \chi \text{ concave increasing, } \chi(0) = 0, \chi(-\infty) = -\infty \right\}$$

denote the sets of convex/concave weights. We say that $\chi \in \mathcal{W}^+_M$ if $\exists M > 0$

$$0 \leq |t\chi'(t)| \leq M|\chi(t)| \qquad \text{for all } t \in \mathbb{R}^-.$$

Definition 2.1.11. *Let $\chi \in \mathcal{W} := \mathcal{W}^- \cup \mathcal{W}^+$. We define the χ-energy of a θ-psh function φ as*

$$E_{\chi,\theta}(\varphi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X (-\chi)(\varphi - V_\theta) \langle T^j \wedge \theta_{\min}^{n-j} \rangle \ \in \,] - \infty, +\infty]$$

with $T = \theta + dd^c\varphi$ and $\theta_{\min} = \theta + dd^c V_\theta$. We set

$$\mathcal{E}_\chi(X, \theta) := \{\varphi \in \mathcal{E}(X, \theta) \mid E_{\chi,\theta}(\varphi) < +\infty\}.$$

We denote by $\mathcal{E}_\chi(X, \alpha)$ the set of positive currents in the class α whose global potential has finite χ-energy.

When $\chi \in \mathcal{W}^-$, [BEGZ10, Proposition 2.8] insures that the χ-energy is non-increasing and for an arbitrary θ-psh function φ,

$$E_{\chi,\theta}(\varphi) := \sup_{\psi \geq \varphi} E_{\chi,\theta}(\psi) \in]-\infty, +\infty]$$

over all $\psi \geq \varphi$ with minimal singularities. On the other hand, if $\chi \in \mathcal{W}_M^+$, we loose monotonicity of the χ-energy function but it has been shown in [GZ07, p.465] that

$$\varphi \in \mathcal{E}_\chi(X, \alpha) \qquad \textit{iff} \qquad \sup_{\psi \geq \varphi} E_{\chi,\theta}(\psi) < +\infty$$

over all ψ with minimal singularities. Recall that for all weights $\chi \in \mathcal{W}^-$, $\tilde{\chi} \in \mathcal{W}^+$, we have

$$\mathcal{E}_{\tilde{\chi}}(X, \alpha) \subset \mathcal{E}^1(X, \alpha) \subset \mathcal{E}_\chi(X, \alpha) \subset \mathcal{E}(X, \alpha).$$

For any $p > 0$, we use the notation

$$\mathcal{E}^p(X, \theta) := \mathcal{E}_\chi(X, \theta), \text{ when } \chi(t) = -(-t)^p.$$

2.2 Bimeromorphic images of energy classes

From now on X and Y denote arbitrary n-dimensional compact Kähler manifolds. We recall that a bimeromorphic map $f : X - - > Y$ can be decomposed as

$$\begin{array}{ccc} & \Gamma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & Y \end{array}$$

where π_1, π_2 are two holomorphic and bimeromorphic maps and Γ denotes a desingularization of the graph of f. For any positive closed $(1,1)$-current T on X we set

$$f_\star T := (\pi_2)_\star \pi_1^\star T.$$

For any positive closed (p,p)-current S is not always possible to define the push forward under a bimeromorphic map. However we define $f_\star\langle S\rangle$ in the usual sense in the Zariski open set V where $f : U \to V$ is a biholomorfism and extending to zero in $Y \setminus V$.

2.2.1 Bimeromorphic invariance of the non-pluripolar product

The goal of this section is to show that the non pluripolar product is a bimeromorphic invariant.

Theorem 2.2.1. *Let* $f : X - - > Y$ *be a bimeromorphic map. Let* $\alpha_1, \cdots, \alpha_p \in H^{1,1}(Y, \mathbb{R})$ *be big classes and fix* T_j *be a positive closed* $(1,1)$*-current in* α_j*. Then*

$$f_\star \langle T_1 \wedge \cdots \wedge T_p \rangle = \langle f_\star T_1 \wedge \cdots \wedge f_\star T_p \rangle. \tag{2.2.1}$$

Proof. By definition of a bimeromorphic map, f induces an isomorphism between Zariski open subsets U and V of X and Y, respectively. By construction the non-pluripolar product does not charge pluripolar sets, thus it is enough to check (2.2.1) on V. Since f induces an isomorphism between U and V we have

$$\left(f_\star \langle T_1 \wedge \cdots \wedge T_p \rangle\right)|_V = f_\star \left(\langle T_1 \wedge \cdots \wedge T_p \rangle|_U\right) = f_\star \langle T_1|_U \wedge \cdots \wedge T_p|_U \rangle$$

and

$$\langle f_\star T_1 \wedge \cdots \wedge f_\star T_p \rangle|_V = \langle f_\star(T_1|_U) \wedge \cdots \wedge f_\star(T_p|_U) \rangle.$$

Now, let ω be a Kähler form on X. Upon adding a multiple of ω to each T_j we can assume that their cohomology classes are Kähler. Thus we can find Kähler forms ω_j such that $T_j = \omega_j + dd^c \varphi_j$. Fix $p \in U$ and take a small open set B such that $p \in B \subset U$. In the open set B we can write $\omega_j = dd^c \psi_j$ so that $T_j = dd^c u_j$ on B with $u_j := \psi_j + \varphi_j$. We infer that

$$f_\star \langle \bigwedge_{j=1}^{p} dd^c u_j \rangle = \langle f_\star(dd^c u_1) \wedge \cdots \wedge f_\star(dd^c u_p) \rangle.$$

Indeed on the plurifine open subset $O_k := \bigcap_j \{u_j > -k\}$ we have

$$\begin{aligned} f_\star \left(\mathbf{1}_{O_k} \langle \bigwedge_j dd^c u_j \rangle \right) &= f_\star \left(\mathbf{1}_{O_k} \bigwedge_j dd^c \max(u_j, -k) \right) \\ &= \mathbf{1}_{\bigcap_j \{u_j \circ f^{-1} > -k\}} \bigwedge_j f_\star(dd^c \max(u_j, -k)) \end{aligned}$$

where the last equality follows from the fact that for any positive $(1,1)$-current S with locally bounded potential $(f_\star S)^n = f_\star(S^n)$. □

2.2.2 Condition (V)

Finite energy classes are in general not preserved by bimeromorphic maps (see Example 2.1.7). We introduce a natural condition to circumvent this problem.

Definition 2.2.2. *Fix α a big class on X. Let $\mathcal{T}_\alpha(X)$ denote the set of positive closed $(1,1)$-currents in α. We say that Condition (*`V`*) is satisfied if*

$$f_\star\Big(\mathcal{T}_\alpha(X)\Big) = \mathcal{T}_{f_\star\alpha}(Y)$$

where $\mathcal{T}_{f_\star\alpha}(Y)$ is the set of positive currents in the image class $f_\star\alpha$.

Theorem A of the introduction is a consequence of Theorem 2.2.1 and Proposition 2.2.3.

Proposition 2.2.3. *Fix $\alpha \in H^{1,1}_{big}(X,\mathbb{R})$. If Condition (*`V`*) holds, then*

(i) $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha)$,

(ii) $f_\star(\mathcal{E}(X,\alpha)) = \mathcal{E}(Y, f_\star\alpha)$,

(iii) $f_\star(\mathcal{E}_\chi(X,\alpha)) = \mathcal{E}_\chi(Y, f_\star\alpha)$ *for any weigth* $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$.

Observe that in general $\mathrm{vol}(\alpha) \leq \mathrm{vol}(f_\star\alpha)$ (see Example 2.1.7).

Proof. Fix $T_{\min}$ a current with minimal singularities in α. Observe that Condition (`V`) implies that $f_\star T_{\min}$ is still a current with minimal singularities, thus

$$\mathrm{vol}(\alpha) = \int_X \langle T^n_{\min}\rangle = \int_Y \langle (f_\star T_{\min})^n\rangle = \mathrm{vol}(f_\star\alpha).$$

Fix $T \in \mathcal{T}_\alpha(X)$. Using Theorem 2.2.1, the change of variables formula and the fact that the pluripolar product does not put mass on analytic sets we get

$$\int_X \langle T^n\rangle = \int_Y \langle (f_\star T)^n\rangle$$

hence by (i) it follows that

$$T \in \mathcal{E}(X,\alpha) \Longleftrightarrow f_\star T \in \mathcal{E}(Y, f_\star\alpha).$$

We now want to prove (iii). Let $T = \theta + dd^c\varphi$ and $T_k = \theta + dd^c\varphi^k$ where $\varphi^k = \max(\varphi, V_\theta - k)$ are the canonical approximant (note they have minimal singularities and decrease to φ). We recall that f induces an isomorphism between Zariski opens subsets U and V, thus by (ii) and the change of variables we get that for any $j = 0, \cdots, n$

$$\int_X (-\chi)(\varphi^k - V_\theta)\langle T^j_k \wedge {\theta_{\min}}^{n-j}\rangle = \int_U (-\chi)(\varphi - V_\theta)\langle T^j_k \wedge \theta^{n-j}_{\min}\rangle$$
$$= \int_V (-\chi)(\varphi^k \circ f^{-1} - V_\theta \circ f^{-1})\langle (f_\star T_k)^j \wedge (f_\star\theta_{\min})^{n-j}\rangle$$

hence the conclusion. □

Condition (V) is easy to understand when f is a blow up with smooth center:

Remark 2.2.4. Let $\pi : X \to Y$ be a blow up with smooth center $\mathcal{Z}$, let $E = \pi^{-1}(\mathcal{Z})$ be the exceptional divisor and fix a big class x on X. There exists a unique $\gamma \in \mathbb{R}$ such that at the level of cohomology classes $\alpha_X = \pi^\star \pi_\star \alpha_X + \gamma\{E\}$. Furthermore, for any $(1,1)$-current $S \in \alpha_X$ there exists a $(1,1)$-current $T \in \pi_\star \alpha_X$ such that $S = \pi^\star T + \gamma[E]$ and S is positive iff T is positive and $\gamma \geq -\nu(T, \mathcal{Z})$ (consequence of Proposition 8.16 in [Dem09] together with Corollary 1.1.8 in [Bou02a]). If Condition (V) holds, then any current $S_{\min}$ with minimal singularities in α_X admits the following decomposition

$$S_{\min} = \pi^\star T_{\min} + \gamma[E]$$

where $T_{\min}$ is a current with minimal singularities in $\pi_\star \alpha_X$. When $\gamma \geq 0$, Condition (V) is always satisfied. On the other side, when $\gamma < 0$ this is not necessarily the case since it could happen that for some positive current T in $\pi_\star \alpha_X$, $\nu(T, \mathcal{Z}) < -\gamma$ (see Example 2.1.7 where $\gamma = -\varepsilon$ and $\nu(\omega_{FS}, \mathcal{Z}) = 0$). We observe indeed that Condition (V) is equivalent to require that every current $T_Y \in \pi_\star \alpha_X$ is such that $\nu(T_Y, \mathcal{Z}) \geq -\gamma$.
As the first statement of Proposition 2.2.3 shows, there is a link between Condition (V) and the invariance of the volume under push forward. For example, if $\mathcal{Z} \not\subseteq X \setminus \mathrm{Amp}\,(\pi_\star \alpha_X)$ then

$$\mathrm{vol}(\alpha_X) = \mathrm{vol}(\pi_\star \alpha_X) \Longleftrightarrow \pi_\star\Big(\mathcal{T}_{\alpha_X}(X)\Big) = \mathcal{T}_{\pi_\star \alpha_X}(Y).$$

Indeed ($\Longrightarrow$) is an easy consequence of the fact that under the assumption on the volumes we can decompose any current with minimal singularities $S_{\min} \in \alpha_X$ as $S_{\min} = \pi^\star T + \gamma[E]$ whith $T \in \mathcal{E}(Y, \pi_\star \alpha_X)$. Proposition 2.1.9 implies $\nu(T, \mathcal{Z}) = 0$, hence $\gamma \geq 0$. Let us stress that the assumption on $\mathcal{Z}$ could be removed if we knew that $\nu(T, y) = \nu(T_{\min}, y)$ for any T with full Monge-Ampère mass, for any $T_{\min}$ with minimal singularities in $\pi_\star \alpha_X$ and for any $y \in Y$. It is however quite delicate to get such information at points y which lie outside the ample locus.

Proposition 2.2.5. *Let $f : X - -- > Y$ a bimeromorphic map between compact Kähler manifold of complex dimension 2. Then the following are equivalent:*

(i) $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star \alpha)$

(ii) $f_\star\Big(\mathcal{T}_\alpha(X)\Big) = \mathcal{T}_{f_\star \alpha}(Y)$.

Proof. Let us recall that (ii) always implies (i). Furthermore by Noether's factorization theorem it suffices to consider the case of a blow-up at one point p. We write $\alpha = \pi^\star \pi_\star \alpha + \gamma\{E\}$. We recall that if $\gamma \geq 0$ there is nothing to prove, we can thus assume $\gamma < 0$. Let S be a current with minimal singularities representing α and T a current with minimal singularities representing $\pi_\star\alpha$. By [BEGZ10, Proposition 1.12], $\pi^* T \in \pi^\star\pi_\star\alpha$ is also with minimal singularities. Note that $\pi^\star T$ is cohomologous to $S - \gamma[E]$. Since α is big, the Siu decomposition of S gives in cohomology the Zariski decomposition of α, and similarly the Siu decomposition of $\pi^\star T$ gives the Zariski decomposition of $\pi^\star\pi_\star\alpha$ (see e.g. [Bou04]). Furthermore, since $\pi^\star T$ is minimal every divisor appearing in the singular part of the Siu decomposition of $\pi^\star T$ also appears in the singular part of the Siu decomposition of $S - \gamma[E]$ with larger or equal coefficients. Then we write the Siu decomposition of S and of $\pi^\star T$ as

$$S = \theta + \sum_{i=1}^{N} \lambda_i[D_i] + \lambda_0[E], \quad \pi^\star T = \tau + \sum_{i=1}^{N} \eta_i[D_i] + \eta_0[E]$$

with $D_i \neq E$ for all i, $\lambda_i > 0$, $\lambda_0, \eta_i, \eta_0 \geq 0$, where in particular $\eta_0 = \nu(\pi^\star T, E) = \nu(T, p)$. Moreover $\{\theta\}, \{\tau\}$ are big and nef classes and $\rho_i = \lambda_i - \eta_i \geq 0$, $\rho_0 = \lambda_0 - \gamma - \eta_0 \geq 0$. It follows that

$$\{\theta + A\} = \{\tau\} \tag{2.2.2}$$

where $A = \sum_{i=1}^{N} \rho_i[D_i] + \rho_0[E]$ is an effective $\mathbb{R}$ divisor. Observe that if we show $\rho_0 = 0$ then $\lambda_0 = \eta_0 + \gamma = \nu(T, p) + \gamma \geq 0$ and so we are done. Intersecting first with θ and then with τ the relation (2.2.2), using the assumption on the volumes, i.e. $\{\theta\}^2 = \{\tau\}^2$, the fact that A is effective, and that τ and θ are nef, we find $\{\tau\} \cdot \{A\} = \{\theta\} \cdot \{A\} = 0$. If we develop the square of the left hand side of (2.2.2) we conclude $\{A\}^2 = 0$. Since $\{\theta\}^2 > 0$, the Hodge index theorem shows that $\{A\} = 0$ and since A is effective, it is the zero divisor. Hence $\rho_0 = 0$. □

We expect that $\nu(T, x) = \nu(T_{\min}, x)$ for all $x \in X$ whenever $T \in \mathcal{E}(X, \alpha)$. We show the following partial result in this direction:

Proposition 2.2.6. *Let X be a compact Kähler surface, α be a big class on X and $T \in \mathcal{E}(X, \alpha)$. Then the set $\{x \mid \nu(T, x) > \nu(T_{\min}, x)\}$ is at most countable.*

Proof. We write the Siu decomposition of the current T as $T = R + \sum_{j=1}^{N} \lambda_i[D_i]$. Note that the set $E_+(T) := \{x \in X \mid \nu(T, x) > 0\}$ contains at most finitely many divisors (Proposition 2.1.9). We claim that $\{R\}$ is big and nef. Indeed, by construction the current R has not positive Lelong number along curves and so any current with minimal singularities $R_{\min} \in \{R\}$ has the same property. Thus the Zariski decomposition of $\{R\}$ is of the type $\{R\} = \{R\} + 0$. Furthermore

$$\mathrm{vol}(\{R\}) \leq \mathrm{vol}(\alpha) = \int_X \langle T^2 \rangle = \int_X \langle R^2 \rangle \leq \mathrm{vol}(\{R\}),$$

that implies $\mathrm{vol}(\alpha) = \{R\}^2 > 0$. Then $T = R + \sum_{j=1}^N \rho_i[D_i] + \sum_{j=1}^N \eta_i[D_i]$, where $\eta_i = \nu(T_{\min}, D_i)$ with $T_{\min} \in \alpha$. Clearly $\rho_i \geq 0$, for any i. We want to show that $\rho_i = 0$. Set $S := R + \sum_{j=1}^N \rho_i[D_i]$ and write the Zariski decomposition of α as $\alpha = \alpha_1 + \sum_{j=1}^N \eta_i\{D_i\}$. Then $\alpha_1 = \{S\}$. This means that $\{S\}$ is big and nef and $\mathrm{vol}(\alpha) = \alpha_1^2 = \{S\}^2$. Now, $\{R + A\} = \{S\}$ where $A = \sum_{j=1}^N \rho_i[D_i]$ is an effective $\mathbb{R}$ divisor. Using the same arguments in the proof of Proposition 2.2.5 we get $\{A\} \cdot \{R\} = \{A\} \cdot \{S\} = \{A\}^2 = 0$ and using the Hodge index theorem we conclude. □

2.3 Sums of finite energy currents

Let X be a compact Kähler manifold of complex dimension n and let α and β be big classes on X. Given two positive currents $T \in \alpha$ and $S \in \beta$ with full Monge-Ampère mass, it is natural to wonder whether $T + S$ has full Monge-Ampère mass in $\alpha + \beta$, and conversely.

2.3.1 Stability of energy classes

We start proving Theorem B of the introduction.

Theorem 2.3.1. *Fix $T \in \mathcal{T}_\alpha(X)$, $S \in \mathcal{T}_\beta(X)$ and $\chi \in \mathcal{W}^- \cup \mathcal{W}_M^+$. Then*

(i) $T + S \in \mathcal{E}(X, \alpha + \beta)$ implies $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$,

(ii) $T + S \in \mathcal{E}_\chi(X, \alpha + \beta)$ implies $T \in \mathcal{E}_\chi(X, \alpha)$ and $S \in \mathcal{E}_\chi(X, \beta)$.

If α, β are Kähler classes, then conversely

(iii) $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$ implies $T + S \in \mathcal{E}(X, \alpha + \beta)$,

(iv) $T \in \mathcal{E}_\chi(X, \alpha)$ and $S \in \mathcal{E}_\chi(X, \beta)$ implies $T + S \in \mathcal{E}_\chi(X, \alpha + \beta)$.

Proof. Pick θ_α and θ_β smooth representatives in α and β, so that $\tilde{\theta} := \theta_\alpha + \theta_\beta$ is a smooth form representing $\alpha + \beta$. We decompose $T = \theta_\alpha + dd^c\varphi$ and $S = \theta_\beta + dd^c\psi$. We assume $\varphi + \psi \in \mathcal{E}(X, \tilde{\theta})$, and first prove that φ has full mass, which is equivalent to showing

$$m_k := \int_{\{\varphi \leq \varphi_{\min} - k\}} \langle (\theta_\alpha + dd^c \max(\varphi, \varphi_{\min} - k))^n \rangle \longrightarrow 0 \qquad as \quad k \to +\infty$$

where $T_{\min} = \theta_\alpha + dd^c\varphi_{\min}$ has minimal singularities in α ([BEGZ10, p.229]). First, observe that on $X \setminus \{\psi = -\infty\}$ we have

$$\{\varphi \leq \varphi_{\min} - k\} \subseteq \{\varphi + \psi \leq \varphi_{\min} + \psi - k\} \subseteq \{\varphi + \psi \leq \phi_{\min} - k\}$$

where $S_{\min} = \tilde{\theta} + dd^c\phi_{\min}$ has minimal singularities in $\alpha + \beta$. Since the non-pluripolar product does not charge pluripolar sets, we infer

$$\begin{aligned} 0 \le m_k \ &\le \int_{\{\varphi+\psi\le\phi_{\min}-k\}} \langle(\theta_\alpha + dd^c \max(\varphi, \varphi_{\min} - k))^n\rangle \\ &\le \int_{\{\varphi+\psi\le\phi_{\min}-k\}\setminus\{\psi=-\infty\}} \langle(\tilde{\theta} + dd^c \max(\varphi+\psi, \varphi_{\min} + \psi - k))^n\rangle \\ &\le \int_{\{\varphi+\psi\le\phi_{\min}-k\}} \langle(\tilde{\theta} + dd^c \max(\varphi+\psi, \phi_{\min} - k))^n\rangle \end{aligned}$$

where the last inequality follows from the fact that $\phi_{\min}$ is less singular then $\varphi_{\min} + \psi$ (see [BEGZ10, Proposition 2.14]). But, by assumption, the last term goes to 0 as k tends to $+\infty$, hence the conclusion. Changing the role of φ and ψ one can prove similarly that also ψ is with full Monge-Ampère mass.

We now prove the second statement. By assumption $\varphi + \psi \in \mathcal{E}_\chi(X, \tilde{\theta})$ with χ a convex weight and so from above we know that φ and ψ both have full Monge-Ampère mass. It suffices to check that $\varphi \in \mathcal{E}_\chi(X, \theta_\alpha)$. By [BEGZ10],

$$E_{\chi,\theta}(\varphi) < +\infty \quad \textit{iff} \quad \sup_k \int_X (-\chi)(\varphi_k - \varphi_{\min}) MA(\varphi_k) < +\infty,$$

for any sequence φ_k of θ_α-psh functions with full Monge-Ampère mass decreasing to φ. Since $T_1 \le T_2$ implies $\langle T_1^n\rangle \le \langle T_2^n\rangle$ we obtain

$$\begin{aligned} &\int_X (-\chi)(\varphi_k - \varphi_{\min})\langle(\theta_\alpha + dd^c\varphi_k)^n\rangle \\ &\qquad \le \int_{X\setminus\{\psi=-\infty\}} (-\chi)(\varphi_k - \varphi_{\min})\langle(\tilde{\theta} + dd^c(\varphi_k + \psi))^n\rangle \\ &\qquad \le \int_{X\setminus\{\psi=-\infty\}} (-\chi)(\varphi_k + \psi - \phi_{\min})\mathrm{MA}\,(\varphi_k + \psi) \end{aligned}$$

where the last inequality follows from monotonicity of χ and the fact that on $X \setminus \{\psi = -\infty\}$

$$\varphi_k - \varphi_{\min} = (\varphi_k + \psi) - (\varphi_{\min} + \psi) \ge (\varphi_k + \psi) - \phi_{\min}.$$

Therefore $E_{\chi,\tilde{\theta}}(\varphi + \psi) < +\infty$ implies $E_{\chi,\theta_\alpha}(\varphi) < +\infty$, as desired.

Assume now that α, β are both Kähler classes and choose Kähler forms $\omega_\alpha \in \alpha$, $\omega_\beta \in \beta$ as smooth representatives. We want to prove that if $\varphi \in \mathcal{E}(X, \omega_\alpha)$ and $\psi \in \mathcal{E}(X, \omega_\beta)$ then $\varphi + \psi \in \mathcal{E}(X, \omega_\alpha + \omega_\beta)$. Let ω be another Kähler form on X. We first show that $\varphi \in \mathcal{E}(X, \omega_\alpha)$ (resp. $\varphi \in \mathcal{E}_\chi(X, \omega_\alpha)$) if and only if $\varphi \in \mathcal{E}(X, \omega)$ (resp. $\varphi \in \mathcal{E}_\chi(X, \omega)$) whenever $\varphi \in PSH(X, \omega)$. We recall that, since ω_α and ω are Kähler forms, there exists a constant $C > 0$ such that $\frac{1}{C}\omega \leq \omega_\alpha \leq C\omega$. Thus

$$\begin{aligned}\int_{\{\varphi\leq -k\}}(\omega_\alpha + dd^c\varphi_k)^n &\leq \int_{\{\varphi\leq -k\}}(C\omega + dd^c\varphi_k)^n \\ &\leq \tilde{C}\sum_{j=0}^{n}\int_{\{\varphi\leq -k\}}\omega^j\wedge(\omega + dd^c\varphi_k)^{n-j},\end{aligned}$$

where $\varphi_k := \max(\varphi, -k)$. And so $\varphi \in \mathcal{E}(X,\omega)$ implies $\varphi \in \mathcal{E}(X,\omega_\alpha)$. Analogously one can prove the reverse. Similarly, for any weight $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$,

$$\int_X -\chi(\varphi_k)(\omega_\alpha + dd^c\varphi_k)^n \leq \tilde{C}\sum_{j=0}^{n}\int_X -\chi(\varphi_k)(\omega + dd^c\varphi_k)^j\wedge\omega^{n-j}.$$

Thus, if $\varphi \in \mathcal{E}_\chi(X,\omega)$ then $\varphi \in \mathcal{E}_\chi(X,\omega_\alpha)$. With the same argument we get the reverse. Now, let ω be a Kähler form such that $\omega_\alpha, \omega_\beta \leq \omega$. From above we have that $\varphi, \psi \in \mathcal{E}(X,\omega)$ (resp. $\varphi, \psi \in \mathcal{E}_\chi(X,\omega)$) and since the energy classes are convex ([GZ07, Propositions 1.6, 2.10 and 3.8]), it follows $\varphi + \psi \in \mathcal{E}(X, 2\omega)$ (resp. $\varphi + \psi \in \mathcal{E}_\chi(X, 2\omega)$). From the previous observation we can deduce $\varphi + \psi \in \mathcal{E}(X, \omega_\alpha + \omega_\beta)$. □

Examples 2.3.5 and 2.3.7 below show the reverse implication is not true in general. This is particularly striking if the following condition is not satisfied:

Definition 2.3.2. *We say that pseudoeffective classes $\alpha_1, \cdots, \alpha_p$ satisfy Condition $\mathcal{MS}$ if the sum $T_1 + \cdots + T_p$ of positive currents $T_i \in \alpha_i$ with minimal singularities has minimal singularities in $\alpha_1 + \cdots + \alpha_p$.*

Note that if $\alpha_1, \cdots, \alpha_p$ satisfy Condition $\mathcal{MS}$ the positive intersection class $\langle\alpha_1 \cdots \alpha_p\rangle$ turns to be multi-linear while it is not so in general ([BEGZ10, p.219]).

Proposition 2.3.3. *Let $T \in \mathcal{T}_\alpha(X)$ and $\chi \in \mathcal{W}^- \cup \mathcal{W}^-_M$. Assume that α is a Kähler class and β is a semi-positive class. Fix $\theta_\beta \in \beta$ a semipositive form. Then*

(i) $T + \theta_\beta \in \mathcal{E}(X, \alpha + \beta)$ if and only if $T \in \mathcal{E}(X, \alpha)$,

(ii) $T + \theta_\beta \in \mathcal{E}_\chi(X, \alpha + \beta)$ if and only if $T \in \mathcal{E}_\chi(X, \alpha)$.

We will exhibit an Example 2.3.5 such that α is semipositive, β is Kähler, θ_β is a Kähler form in β, $T \in \mathcal{E}^1(X,\alpha)$ but $T + \theta_\beta \notin \mathcal{E}^1(X, \alpha + \beta)$.

Proof. We will first prove the second statement. Fix ω, θ_β smooth representatives of α and β, respectively and denote $\tilde{\omega} := \omega + \theta_\beta$. Note that ω can be chosen to be Kähler. Let $T := \omega + dd^c\varphi \in \mathcal{E}_\chi(X,\alpha)$, by [BEGZ10] we have

$$E_{\chi,\omega}(\varphi) \Longleftrightarrow \sup_k E_{\chi,\omega}(\varphi_k) < +\infty$$

where $\varphi_k := \max(\varphi, -k)$. We now show that $E_{\chi,\tilde\omega}(\varphi_k)$ is uniformly bounded from above. Fix A such that $\tilde\omega \le (A+1)\omega$. Then

$$\begin{aligned}&\int_X -\chi(\varphi_k)\,(\tilde\omega + dd^c\varphi_k)^j \wedge \tilde\omega^{n-j}\\ &\le (A+1)^{n-j}\int_X -\chi(\varphi_k)\,(A\omega+\omega+dd^c\varphi_k)^j\wedge\omega^{n-j}\\ &\le C\sum_{l=0}^{j}\int_X -\chi(\varphi_k)\,(\omega+dd^c\varphi_k)^{j-l}\wedge\omega^{n-j+l} \le C'\,E_{\chi,\omega}(\varphi_k).\end{aligned}$$

The first statement is an easy consequence of the second one recalling that

$$\mathcal{E}(X,\alpha) = \bigcup_{\chi\in\mathcal{W}^-}\mathcal{E}_\chi(X,\alpha).$$

The reverse inclusions is Theorem 2.3.1. □

Remark 2.3.4. Let us stress that the first statement of Proposition 2.3.3 could be proved in great generality (α, β big classes such that Condition $\mathcal{MS}$ holds, θ_β current with minimal singularities) if given $\alpha_1,\cdots,\alpha_n$ big classes and $T_1 \in \mathcal{E}(X,\alpha_1)$, the following would hold

$$\int_X \langle T_1\wedge\theta_{2,\min}\wedge\ldots\wedge\theta_{n,\min}\rangle = \int_X\langle\theta_{1,\min}\wedge\ldots\wedge\theta_{n,\min}\rangle$$

where $\theta_{i,\min} := \theta_i + dd^cV_{\theta_i}\in\alpha_i$.

2.3.2 Counterexamples

The following example shows that given two currents $T\in\mathcal{E}^1(X,\alpha)$ and $S\in\mathcal{E}^1(X,\beta)$ we can not expect that $T+S\in\mathcal{E}^1(X,\alpha+\beta)$, even if α is semipositive and β is Kähler.

Example 2.3.5. Let $\pi : X\to\mathbb{P}^2$ be the blow up at one point p and set $E := \pi^{-1}(p)$. Fix $\alpha = \pi^\star\{\omega_{FS}\}$ and $\beta = 2\pi^\star\{\omega_{FS}\} - \{E\}$ so that $\alpha+\beta = 3\pi^\star\{\omega_{FS}\}-\{E\}$. We pick $\tilde\omega\in\alpha+\beta$ a Kähler form of the type $\tilde\omega = \pi^\star\omega_{FS}+\omega$, where $\omega\in\beta$ is a Kähler form. We will show that

$$\mathcal{E}^1(X,\alpha)\nsubseteq\mathcal{E}^1(X,\alpha+\beta)\cap\mathcal{T}_\alpha(X).$$

The goal is to find a ω_{FS}-psh function φ on $\mathbb{P}^2$ such that $\pi^\star\varphi\in\mathcal{E}^1(X,\pi^\star\omega_{FS})$ but $\pi^\star\varphi\notin\mathcal{E}^1(X,\tilde\omega)$. Let U be a local chart of $\mathbb{P}^2$ such that $p\to(0,0)\in U$. We define

$$\varphi_\delta := \frac{1}{C}\chi\cdot u_\delta - K_\delta$$

where $u_\delta := -(-\log\|z\|)^\delta$, χ is a smooth cut-off function such that $\chi\equiv 1$ on $\mathbb{B}$ and $\chi\equiv 0$ on $U\setminus\mathbb{B}(2)$, K_δ is a positive constant such that $\varphi_\delta\le -1$

and $C > 0$. Choosing C big enough φ_δ induces a ω_{FS}-psh function on $\mathbb{P}^2$, say $\tilde{\varphi}_\delta$. Note that by [CGZ08, Corollary 2.6] $\tilde{\varphi}_\delta \in \mathcal{E}(\mathbb{P}^2, \omega_{FS})$ if $0 \leq \delta < 1$. We let the reader check that $\tilde{\varphi}_\delta \in W^{1,2}(\mathbb{P}^2, \omega_{FS})$ for all $0 \leq \delta < 1$. Therefore $\tilde{\varphi}_\delta \in \mathcal{E}^1(\mathbb{P}^2, \omega_{FS})$ iff

$$\int_{\mathbb{P}^2} -\tilde{\varphi}_\delta (dd^c \tilde{\varphi}_\delta)^2 < +\infty$$

We claim this is the case iff $0 \leq \delta < \frac{2}{3}$.
Note that $\tilde{\varphi}_\delta$ is smooth outside p, therefore we have to check that

$$\int_{\mathbb{B}(\frac{1}{2})} -u_\delta (dd^c u_\delta)^2 < +\infty. \tag{2.3.1}$$

Set $\chi(t) = -(-t)^\delta$ so that $u_\delta = \chi(\log \|z\|)$. Then on $\mathbb{B}(\frac{1}{2}) \setminus \{(0,0)\}$ we have

$$(dd^c u_\delta)^2 = C_1 \frac{1}{8\|z\|^4} \chi^{''} \cdot \chi^{'}(\log \|z\|) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$

hence the convergence of the integral in (2.3.1) is equivalent to the convergence of

$$\int_{\mathbb{B}(\frac{1}{2})\setminus\{(0,0)\}} \frac{-\chi(\log \|z\|) \cdot \chi^{''}(\log \|z\|) \cdot \chi^{'}(\log \|z\|)}{\|z\|^4} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$

$$= \int_0^{\frac{1}{2}} \frac{-\chi(\log \rho) \cdot \chi^{''}(\log \rho) \cdot \chi^{'}(\log \rho)}{\rho} d\rho = \delta(1-\delta) \int_{-\log \frac{1}{2}}^{+\infty} \frac{1}{(s)^{3-3\delta}} ds$$

which is finite iff $0 \leq \delta < \frac{2}{3}$, as claimed. Therefore by Proposition 2.2.3 we get $\pi^\star \tilde{\varphi}_\delta \in \mathcal{E}^1(X, \pi^\star \omega_{FS})$. But $\pi^\star \tilde{\varphi}_\delta \notin \mathcal{E}^1(X, \tilde{\omega})$ if $\frac{1}{2} \leq \delta < \frac{2}{3}$ since

$$|\nabla(\pi^\star \tilde{\varphi}_\delta)| \notin L^2(X, (\tilde{\omega})^2) \quad \textit{if} \quad \delta \geq \frac{1}{2}.$$

Indeed, let $z = (z_1, z_2) \in \mathbb{B}$ and fix a coordinate chart in X, then $\pi(s,t) = (z_1, z_2) = (s, st)$. Therefore, on $\pi^{-1}(\mathbb{B})$

$$\varphi_\delta \circ \pi(s,t) = \frac{1}{C} u_\delta(s, st) = -\frac{1}{C}\left(-\log|s| - \log\sqrt{1+|t|^2}\right)^\delta$$

Hence,

$$\int_{\pi^{-1}(\mathbb{B})} \left|\frac{\partial(\varphi_\delta \circ \pi)}{\partial s}\right|^2 ds \wedge d\bar{s} \wedge dt \wedge d\bar{t} \geq \left(\frac{\delta}{2C}\right)^2 \int_{\pi^{-1}(\mathbb{B})} \frac{ds \wedge d\bar{s} \wedge dt \wedge d\bar{t}}{|s|^2(-\log|s|)^{2-2\delta}}$$

which is not finite if $\delta \geq \frac{1}{2}$. The conclusion follows from [GZ07, Theorem 3.2].

Remark 2.3.6. Observe that α, β satisfy Condition $\mathcal{MS}$ in previous example and also that $\pi^\star\tilde{\varphi}_\delta \in \mathcal{E}(X, \tilde{\omega})$. Indeed, let $T := \pi^\star\omega_{FS} + dd^c(\tilde{\varphi}_\delta \circ \pi)$, we need to check that $T + \omega \in \mathcal{E}(X, \alpha + \beta)$. Since $T \in \mathcal{E}(X, \alpha)$ and

$$\langle (T + \omega)^2 \rangle = \langle T^2 \rangle + 2\langle T \rangle \wedge \omega + (\omega)^2.$$

it suffices to show that

$$\{\langle T \rangle \wedge \omega\} = \{\pi^\star\omega_{FS}\} \cdot \{\omega\}.$$

which is equivalent to

$$\{(T - \langle T \rangle) \wedge \omega\} = 0.$$

Hence, what we need to show is that $T - \langle T \rangle = 0$. The $(1,1)$-current $T - \langle T \rangle$ is positive and is supported by the exceptional divisor E. Therefore using [Dem09, Corollary 2.14] it results that

$$T = \langle T \rangle + \gamma[E]$$

where $\gamma = \nu(T, E) = \nu(\pi_\star T, p) = 0$ since $\delta < 1$. And so the conclusion.

Previous remark could let us think whenever $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$ then $T + S \in \mathcal{E}(X, \alpha + \beta)$, but this is not true either as the following example shows:

Example 2.3.7. Let $\pi : X \to \mathbb{P}^2$ be the blow up at one point p and set $E := \pi^{-1}(p)$. Consider $\alpha = \pi^\star\{\omega_{FS}\} + \{E\}$ and $\beta = 2\pi^\star\{\omega_{FS}\} - \{E\}$. Thus $\alpha + \beta = 3\pi^\star\{\omega_{FS}\}$. Since β is a Kähler class we can choose $S = \omega$ with ω a Kähler form.
Observe that currents with minimal singularities in α are of the type $\pi^\star S_{\min} + [E]$, where $S_{\min}$ is a current with minimal singularities in $\{\omega_{FS}\}$ (Remark 2.2.4). By Lemma 2.1.10

$$\mathrm{vol}(\alpha) = \int_X \langle (\pi^\star S_{\min} + [E])^2 \rangle = \int_X \langle (\pi^\star S_{\min})^2 \rangle = \int_X \pi^\star \langle S_{\min}^2 \rangle = 1,$$

while $\mathrm{vol}(\alpha + \beta) = (\alpha + \beta)^2 = 9$. Let now $T \in \mathcal{E}(X, \alpha)$ and recall that any positive $(1,1)$-current in α is of the form $T = \pi^\star S + [E]$ with $S \in \mathcal{T}_{\{\omega_{FS}\}}(\mathbb{P}^2)$. In particular we choose $T := \pi^\star\omega_{FS} + [E]$. We want to show that $T + \omega \notin \mathcal{E}(X, \alpha + \beta)$. Now, from the multilinearity of the non-pluripolar product we get

$$\int_X \langle (T + \omega)^2 \rangle = \int_X \langle (\pi^\star\omega_{FS} + [E] + \omega)^2 \rangle = \int_X \langle (\pi^\star\omega_{FS} + \omega)^2 \rangle = 8$$

Hence $\int_X \langle (T + \omega)^2 \rangle = 8 < 9 = \mathrm{vol}(\alpha + \beta)$.

The same type of computations show that if we pick $T \in \mathcal{E}(X, \alpha)$, then, for any $0 < \varepsilon \leq 1$, $T + \varepsilon\omega \notin \mathcal{E}(X, \alpha + \varepsilon\omega)$.

Remark 2.3.8. Note that in the latter example α, β do not satisfy Condition $\mathcal{MS}$.

2.4 Comparison of Capacities

Let X be a compact Kähler manifold of complex dimension n and let α be a big class on X. Set $\theta \in$ a smooth form and $\theta_{\min} := \theta + dd^c V_\theta$ the positive $(1,1)$-current in α with 'canonical' minimal singularities.

2.4.1 Intrinsic Capacities

We introduce the space of "$\theta_{\min}$-plurisubharmonic" functions

$$PSH(X, \theta_{\min}) := \{\psi \mid \psi + V_\theta \quad \textit{is a } \theta - \textit{psh function}\}.$$

Note that a $\theta_{\min}$-psh function ψ is not upper-semi-continuous but $\psi + V_\theta$ is.

Monge-Ampère capacity

Following [BEGZ10] we introduce the Monge-Ampère capacity with respect to a big class.

Definition 2.4.1. *We define the capacity of a borel set $K \subseteq X$ as*

$$\mathrm{Cap}_{\theta_{\min}}(K) := \sup\left\{\int_K \langle(\theta_{\min} + dd^c\psi)^n\rangle,\ \psi \in \mathrm{PSH}(X, \theta_{\min}) \mid -1 \le \psi \le 0\right\}.$$

Observe that the above one is the same definition as [BEGZ10, Definition 4.3], just taking $\psi = \varphi - V_\theta$, where φ is a θ-psh function. Here we introduce this equivalent formulation since in Section 2.4 we need the positivity of the reference current $\theta_{\min}$.

The relative extremal function

We introduce the notion of the relative extremal function with respect to $\theta_{\min}$. If E is a Borel subset of X, we set

$$h_{E,\theta_{\min}}(x) := \sup\left\{\psi(x) \,|\, \psi \in PSH(X, \theta_{\min}),\ \psi \le 0 \text{ and } \psi_{|E} \le -1\right\},$$

and

$$h^*_{E,\theta_{\min}} := (h_{E,\theta_{\min}} + V_\theta)^* - V_\theta.$$

It is a standard matter to show that, as in the Kähler case (see [GZ05]), the $\theta_{\min}$-psh function $h^*_{E,\theta_{\min}}$ satisfies

$$\mathrm{Cap}_{\theta_{\min}}(K) = \int_K \mathrm{MA}\,(V_\theta + h^*_{K,\theta_{\min}}) = \int_X (-h^*_{K,\theta_{\min}})\mathrm{MA}\,(V_\theta + h^*_{K,\theta_{\min}})$$

where $K \subset X$ is a compact set (for details see [BBGZ13, Lemma 1.5]).

Capacities of sublevel sets

We now generalize [GZ07, Lemma 5.1].

Lemma 2.4.2. *Fix* $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$*,* $M \geq 1$*. If* $\varphi \in \mathcal{E}_\chi(X,\theta)$*, then*

$$\exists C_\varphi > 0, \forall t > 1,\ \mathrm{Cap}_{\theta_{\min}}(\varphi < V_\theta - t) \leq C_\varphi |t\,\chi(-t)|^{-1}.$$

Conversely if there exists $C_\varphi, \varepsilon > 0$ *such that for all* $t > 1$*,*

$$\mathrm{Cap}_{\theta_{\min}}(\varphi < V_\theta - t) \leq C_\varphi |t^{n+\varepsilon}\,\chi(-t)|^{-1},$$

then $\varphi \in \mathcal{E}_\chi(X,\theta)$*.*

Proof. Fix $\varphi \in \mathcal{E}_\chi(X,\theta)$ and $u \in PSH(X,\theta)$ such that $-1 \leq u - V_\theta \leq 0$. For $t \geq 1$, observe that by [BEGZ10, Proposition 2.14], $\frac{\varphi}{t} + \left(1 - \frac{1}{t}\right) V_\theta \in \mathcal{E}(X,\theta)$ and

$$(\varphi - V_\theta < -2t) \subseteq \left(\frac{\varphi - V_\theta}{t} < -1 + u - V_\theta\right) \subseteq (\varphi - V_\theta < -t).$$

It therefore follows from the generalized comparison principle and from the multilinearity of the non-pluripolar product ([BEGZ10, Propositions 2.2 and 1.4]) that

$$\begin{aligned}\int_{(\varphi - V_\theta < -2t)} MA(u) &\leq \int_{(\varphi - V_\theta < -t)} MA\left(\frac{\varphi}{t} + \left(1 - \frac{1}{t}\right) V_\theta\right) \\ &\leq \left(1 - \frac{1}{t}\right)^n \int_{(\varphi - V_\theta < -t)} \langle \theta^n_{\min} \rangle + t^{-1} \sum_{k=1}^{n} \binom{n}{k} \int_{(\varphi - V_\theta < -t)} \langle T^k \wedge \theta^{n-k}_{\min} \rangle\end{aligned}$$

where $T := \theta + dd^c\varphi$. Furthermore, since

$$MA(V_\theta) = \mathbf{1}_{\{V_\theta = 0\}} \theta^n$$

(see [BD12, Corollary 2.5]), we get

$$\int_{(\varphi - V_\theta < -t)} \langle \theta^n_{\min} \rangle = \int_{(\varphi - V_\theta < -t) \cap D} \theta^n = \mathbf{1}_D \theta^n (\varphi < -t) \leq C \omega^n (\varphi < -t),$$

where $D := \{V_\theta = 0\}$, ω is a Kähler form on X and $C > 0$. We recall that $\mathrm{vol}_\omega(\varphi < -t)$ decreases exponentially fast (see [GZ05]) and observe that for all $1 \leq k \leq n$,

$$\int_{(\varphi - V_\theta < -t)} \langle T^k \wedge \theta^{n-k}_{\min} \rangle \leq \frac{1}{|\chi(-t)|} \int_X (-\chi) \circ (\varphi - V_\theta) \langle T^k \wedge \theta^{n-k}_{\min} \rangle \leq \frac{1}{|\chi(-t)|} E_\chi(\varphi).$$

This yields the first assertion.

The second statement follows from similar arguments as in the Kähler case, working with the θ-psh function $u := \frac{1}{t}\varphi_t + \left(1 - \frac{1}{t}\right) V_\theta$ where $\varphi_t := \max(\varphi, V_\theta - t)$ for any $\varphi \in PSH(X,\theta)$. Let us stress that this is the only place where the assumption on the weight, $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$ is used. □

Alexander capacity

For K a Borel subset of X, we set

$$V_{K,\theta} := \sup\{\varphi \mid \varphi \in PSH(X,\theta),\ \varphi \leq 0 \ \ on\ K\}.$$

Note that

$$V_\theta = V_{X,\theta} \leq V_{K,\theta}$$

by definition. It follows from standard arguments (see [GZ05, Theorem 4.2]) that the usc regularization $V^*_{K,\theta}$ of $V_{K,\theta}$ is either a θ-psh function with minimal singularities (when K is not-pluripolar) or identically $+\infty$ (when K is pluripolar).

Definition 2.4.3 (Alexander-Taylor capacity)**.** *Let K be a Borel subset of X. We set*

$$T_\theta(K) := \exp(-\sup_X V^*_{K,\theta}).$$

As in the Kähler case, the capacities T_θ and $Cap_{\theta_{\min}}$ compares as follows:

Proposition 2.4.4. *There exists $A > 0$ such that for all Borel subsets $K \subset X$,*

$$\exp\left[-\frac{A}{Cap_{\theta_{\min}}(K)}\right] \leq T_\theta(K) \leq e\cdot\exp\left[-\left(\frac{\mathrm{vol}(\alpha)}{\mathrm{Cap}_{\theta_{\min}}(K)}\right)^{\frac{1}{n}}\right]$$

Proof. It suffices to treat the case of compact sets. The second inequality is [BEGZ10, Lemma 4.2]. We prove the first inequality. We can assume that $M := M_\theta(K) \geq 1$ otherwise it is sufficient to adjust the value of A. Let φ be a θ-psh function such that $\varphi \leq 0$ on K. Then $\varphi \leq M$ on X, hence $w := M^{-1}(\varphi - M - V_\theta) \in PSH(X, \theta_{\min})$ satisfies $\sup_X w \leq 0$ and $w \leq -1$ on K. We infer $w \leq h^*_{K,\theta_{\min}}$ and

$$w_K := \frac{V^*_{K,\theta} - M - V_\theta}{M} \leq h^*_{K,\theta_{\min}} \leq 0.$$

Then we get

$$\begin{aligned}
\mathrm{Cap}_{\theta_{\min}}(K) &= \int_X \left(-h^*_{K,\theta_{\min}}\right) \mathrm{MA}\left(V_\theta + h^*_{K,\theta_{\min}}\right)\\
&\leq \frac{1}{M}\int_X -(V^*_{K,\theta} - M - V_\theta)\,\mathrm{MA}\left(V_\theta + h^*_{K,\theta_{\min}}\right)\\
&\leq \frac{C_1}{M}
\end{aligned}$$

with $C_1 > 0$. The last estimate follows from Lemma below together with [GZ05, Proposition 1.7] since $\sup_X(V^*_{K,\theta} - M - V_\theta) = 0$ and by [BD12, Corollary 2.5], $\langle(\theta + dd^c V_\theta)^n\rangle = \mathbf{1}_{\{V_\theta=0\}}\theta^n \leq C\omega^n$. □

The following Lemma is a straightforward generalization of [GZ05, Corollary 2.3], (see also [BBGZ13, Lemma 3.2]).

Lemma 2.4.5. *Let ψ, φ be θ-psh functions with minimal singularities with φ normalized in such a way that $0 \leq \varphi - V_\theta \leq 1$. Then we have*

$$\int_X -(\psi - V_\theta)\langle(\theta + dd^c\varphi)^n\rangle \leq \int_X -(\psi - V_\theta)\langle(\theta + dd^c V_\theta)^n\rangle + n\,\mathrm{vol}(\alpha).$$

2.4.2 Comparing Capacities

We introduce a slighty different notion of big capacity that is comparable with respect to the usual one. For any Borel set $K \subset X$ we define

$$\mathrm{Cap}^\lambda_{\theta_{\min}}(K) := \sup\left\{\int_K \langle(\theta_{\min} + dd^c\psi)^n\rangle,\, \psi \in PSH(X, \theta_{\min}) \,|\, -\lambda \leq \psi \leq 0\right\},$$

where $\lambda \geq 1$. We let the reader check that

$$\mathrm{Cap}_{\theta_{\min}}(K) \leq \mathrm{Cap}^\lambda_{\theta_{\min}}(K) \leq \lambda^n \mathrm{Cap}_{\theta_{\min}}(K). \tag{2.4.1}$$

We now compare the Monge-Ampère capacities w.r.t. different big classes (Theorem D of the introduction).

Theorem 2.4.6. *Let α_1 and α_2 be big classes on X such that $\alpha_1 \leq \alpha_2$. We assume that $\{\alpha_1, \alpha_2 - \alpha_1\}$ satisfies Condition $\mathcal{MS}$ and that there exists a positive $(1,1)$-current $T_0 \in \alpha_2 - \alpha_1$ with bounded potentials. Then there exist $C > 0$ such that for any Borel set $K \subset X$,*

$$\frac{1}{C}\,\mathrm{Cap}_{\theta_{1,\min}}(K) \leq \mathrm{Cap}_{\theta_{2,\min}}(K) \leq C\left(\mathrm{Cap}_{\theta_{1,\min}}(K)\right)^{\frac{1}{n}}.$$

Note that in case of Kähler forms the result is stronger and the proof much simpler (see [BEGZ10, Proposition 2.5]) but we can not expect better in the general case of big classes. In the following, Example 2.4.7 shows that the exponent at the right-hand side is necessary.

Proof. Fix $\theta_1 \in \alpha_1$, $\theta_2 \in \alpha_2$ smooth forms. Write $T_0 = (\theta_2 - \theta_1) + dd^c f_0$ where f_0 is a bounded potential. Let φ be a θ_1-psh function such that $-1 \leq \varphi - V_{\theta_1} \leq 0$ then $\varphi + f_0$ is a θ_2-psh function. Condition $\mathcal{MS}$ insures that the potential $V_{\theta_1} + f_0$ has minimal singularities, thus there exists a positive constant C such that $|V_{\theta_2} - V_{\theta_1} - f_0| \leq C$. Therefore $-\lambda \leq \varphi + f_0 - C - V_{\theta_2} \leq 0$ where $\lambda = 1 + 2C$. Now, using (2.4.1) and the fact that $T_1 \leq T_2$ implies $\langle T_1^n\rangle \leq \langle T_2^n\rangle$ we get

$$\int_K \langle(\theta_1 + dd^c\varphi)^n\rangle \quad \leq \quad \int_K \langle(\theta_2 + dd^c(\varphi + f_0)^n\rangle$$

namely $\mathrm{Cap}_{\theta_{1,\min}}(K) \leq \mathrm{Cap}^{\lambda}_{\theta_{2,\min}}(K) \leq \lambda^n \mathrm{Cap}_{\theta_{2,\min}}(K)$ hence the left inequality. In order to prove the other inequality we have to go through the Alexander capacity. Since $V^*_{\theta_1,K} + f_0 \leq V^*_{\theta_2,K}$

$$\sup_X (V^*_{\theta_2,K}) \geq \sup_X (V^*_{\theta_1,K}) + \inf_X f_0,$$

and so

$$T_{\theta_2}(K) \leq T_{\theta_1}(K) \cdot e^{-\inf_X f_0}.$$

Furthermore, using Proposition 2.4.4 we get

$$\begin{aligned} \exp\left[-\frac{A}{\mathrm{Cap}_{\theta_{2,\min}}(K)}\right] &\leq T_{\theta_2}(K) \\ &\leq T_{\theta_1}(K) \cdot e^{-\inf_X f_0 + 1} \\ &\leq e^{-\inf_X f_0 + 1} \cdot \exp\left[-\left(\frac{\mathrm{vol}(\alpha_1)}{\mathrm{Cap}_{\theta_{1,\min}}(K)}\right)^{\frac{1}{n}}\right] \end{aligned}$$

with A a positive constant. Thus, there exists a constant $C > 0$ such that

$$\begin{aligned} \mathrm{Cap}_{\theta_{2,\min}}(K) &\leq A \left[\left(\frac{\mathrm{vol}(\alpha_1)}{\mathrm{Cap}_{\theta_{1,\min}}(K)}\right)^{\frac{1}{n}} + \inf_X f_0 - 1\right]^{-1} \\ &\leq C\, \mathrm{Cap}_{\theta_{1,\min}}(K)^{\frac{1}{n}}. \end{aligned}$$

Hence the conclusion. □

Example 2.4.7. Let $\pi : X \to \mathbb{P}^2$ the blow-up at one point p and set $E := \pi^{-1}(p)$. Consider $\alpha_1 = \{\pi^\star \omega_{FS}\}$ and $\alpha_2 = \{\tilde{\omega}\}$ where $\tilde{\omega}$ is a Kähler form on X. Let Δ_r be the polydisc of radius $r < 1$ on $\mathbb{P}^2$. By [GZ05, Proposition 2.10] and [Kli91, Lemma 4.5.8] we know that

$$\mathrm{Cap}_{\pi^\star \omega_{FS}}(\pi^{-1}(\Delta_r)) = \mathrm{Cap}_{\omega_{FS}}(\Delta_r) \sim \frac{1}{(-\log r)^2}.$$

Fix now a local chart $U \subset X$ such that $p \in U$ and consider $K_r \subset U$, $K_r := \{(s,t) \in U \mid 0 < \|s\| < r,\ 0 < \|t\| < 1\}$. Then

$$\mathrm{Cap}_{\tilde{\omega}}(\pi^{-1}(\Delta_r)) \geq \mathrm{Cap}_{\tilde{\omega}}(K_r) \sim C \frac{1}{-\log r},$$

with C a positive constant.

2.4.3 Energy classes with homogeneous weights

As Example 2.3.5 shows we can not hope to get stability of weighted energy classes $\mathcal{E}_\chi$ by only adding Condition $\mathcal{MS}$. We nevertheless establish a partial stability property with a gap for energy classes with respect to homogeneous weights $\chi(t) = -(-t)^p$. We recall that the functions $\chi(t) = -(-t)^p$ belong to $\mathcal{W}^-$ if $0 < p \leq 1$ while they belong to $\mathcal{W}^+_M$ when $p \geq 1$.

Proposition 2.4.8. *Let α, β be big classes. Assume that $S \in \beta$ has bounded potential and the couple (α, β) satisfies Condition $\mathcal{MS}$. If $p > n^2 - 1$ then*

$$T \in \mathcal{E}^p(X, \alpha) \Longrightarrow T + S \in \mathcal{E}^q(X, \alpha + \beta),$$

where $0 < q < p - n^2 + 1$.

Proof. Fix $\theta_\alpha, \theta_\beta$ smooth representatives of α, β, respectively and set $\tilde{\theta} := \theta_\alpha + \theta_\beta$. Write $S = \theta_\beta + dd^c\psi$ and denote $\theta_{\alpha,\min} := \theta_\alpha + dd^c V_{\theta_\alpha}$ and $\tilde{\theta}_{\min} := \tilde{\theta} + dd^c V_{\tilde{\theta}}$. We want to show that there exists a positve number $q < p$ such that given a θ_α-psh function $\varphi \in \mathcal{E}^p(X, \theta_\alpha)$ then $\varphi + \psi \in \mathcal{E}^q(X, \tilde{\theta})$. By the first claim of Lemma 2.4.2, for any $t > 1$ there exists a constant $C_\varphi > 0$ such that

$$\mathrm{Cap}_{\theta_{\alpha,\min}}(\varphi - V_{\theta_\alpha} < -t) \leq C_\varphi t^{-(p+1)}. \tag{2.4.2}$$

The goal is to find a similar estimate from above of the quantity $\mathrm{Cap}_{\tilde{\theta}_{\min}}(\varphi + \psi - V_{\tilde{\theta}} < -t)$. Set $K := \{\varphi - V_{\theta_\alpha} < -t\}$ and $\tilde{K} := \{\varphi + \psi - V_{\tilde{\theta}} < -t\}$. We infer that Condition $\mathcal{MS}$ implies $\tilde{K} \subseteq K$. Thus $\mathrm{Cap}_{\tilde{\theta}_{\min}}(\tilde{K}) \leq \mathrm{Cap}_{\tilde{\theta}_{\min}}(K)$. Now, by Theorem 2.4.6 we know that there exists $A > 0$ such that

$$\mathrm{Cap}_{\tilde{\theta}_{\min}}(\tilde{K}) \leq A\, \mathrm{Cap}_{\theta_{\alpha,\min}}(K)^{\frac{1}{n}} \leq \tilde{C}_\varphi\, t^{-\frac{p+1}{n}}$$

where the last inequality follows from (2.4.2). This means that there exist $C_\varphi, \varepsilon > 0$ such that

$$\mathrm{Cap}_{\tilde{\theta}_{\min}}(\tilde{K}) \leq C_\varphi t^{-(n+\varepsilon+q)}$$

with $0 < q < p - n^2 + 1 - n\varepsilon$. Hence by Lemma 2.4.2 we get $\varphi + \psi \in \mathcal{E}^q(X, \tilde{\theta})$. $\square$

Chapter 3

Finite energy measures

Introduction

In [BBGZ13] the authors show that degenerate complex Monge-Ampère equations in a big cohomology class of a compact Kähler manifold can be solved using a variational method independent of Yau's theorem. In particular, they define the *electrostatic energy* $E^*(\mu)$ of a probability measure μ on X which is a pluricomplex analogue of the classical logarithmic energy of a measure.

They then give a very nice and useful caracterization (that for our purposes we will take as definition) of measures μ with finite energy:

Definition. A non-pluripolar probability measure μ has finite energy in a big class α on X if and only if there exists $T \in \mathcal{E}^1(X, \alpha)$ such that

$$\mu = \frac{\langle T^n \rangle}{\mathrm{vol}(\alpha)}.$$

In this case we write $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha))$.

It is natural to wonder if such a notion is a bimeromorphic invariant. It turns out that it is invariant under biholomorphism but not under bimeromorphisms. Similarly we consider the dipendence on the cohomology classes. We prove the following:

Proposition A. Let α, β be Kähler classes. Then

$$\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha)) \Longleftrightarrow \mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \beta)).$$

On the other hand in Example 3.3.2 we show that this notion is not bimeromorphic invariant, and in general it depends on the cohomology class. In fact, we have

Proposition B. Let $\pi : X \to \mathbb{P}^2$ be the blow-up at one point. Then there exists a probability measure μ and a Kähler class $\{\tilde{\omega}\}$ on X such that

(i) $\mu \in \mathrm{MA}\left(\mathcal{E}^1(X,\{\tilde{\omega}\})\right)$ but $\mu \notin \mathrm{MA}\left(\mathcal{E}^1(X,\{\pi^\star \omega_{FS}\})\right)$,

(ii) $\pi_\star \mu \notin \mathrm{MA}\left(\mathcal{E}^1(X,\{\pi_\star \tilde{\omega}\})\right)$.

We then work in the Kähler setting and we give some criteria in order to insure that a given non-pluripolar probability measure has finite energy. Observe that, given a non-pluripolar probability measure μ and a Kähler form ω normalized such that $\mathrm{vol}(\omega) = 1$, by [GZ07] we can always solve the Monge-Ampère equation

$$(\omega + dd^c \varphi)^n = \mu.$$

whith $\varphi \in \mathcal{E}(X,\omega)$. Giving condition for having μ with finite energy is therefore equivalent to establish whenever φ belongs to $\mathcal{E}^1(X,\omega)$.

Following the ideas in [DNL14a, DNL14b], we are able to do that when μ is dominated by the generalized Monge-Ampère capacity:

Proposition C. Let $\psi \in \mathcal{E}^1(X,\omega/2)$. Assume there exists a constant $A > 0$ such that

$$\mu \leq A\,\mathrm{Cap}_\psi^{1+\varepsilon}$$

for some $\varepsilon > 0$. Then μ has finite energy in $\{\omega\}$.

Here Cap_ψ denotes the generalized Monge-Ampère capacity, defined for any Borel set $E \subset X$ as

$$\mathrm{Cap}_\psi(E) := \sup\left\{ \int_E \mathrm{MA}\,(u) \quad | \quad u \in \mathrm{PSH}(X,\omega),\ \ \psi - 1 \leq u \leq \psi \right\}.$$

We also look at measure with densities, i.e. of type $\mu = f dV$, and we ask under which conditions on f we are able to insure $\mu \in \mathrm{MA}\,(\mathcal{E}(X,\alpha))$.

If $f \in L^p(X)$, with $p > 1$, then from Kołodziej's work we can deduce that μ is the Monge-Ampère measure of a bounded function and in particular has finite energy. Clearly, the assumption of being in L^p for some $p > 1$ is very strong and we search for some "minimal" conditions on f.

We consider, for example, densities that locally can be written as

$$f = \frac{h}{\prod_{j=1}^n |z_j|^2(-\log|z_j|)^{1+\alpha}}$$

where h is a smooth function, $1/B \leq h \leq B$ for some $B > 0$ and $\alpha > 0$. Let us stress that in this case $f \in L^1(X)$ but not in $L^p(X)$ for any $p > 1$. We can nevertheless give a complete caracterization in these special cases:

Proposition D. Let ω be a Kähler form. The following holds:

(i) If $\alpha > 1/2$, then $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\{\omega\}))$.

(ii) If $\alpha \leq 1/2$, then $\mu \notin \mathrm{MA}\left(\mathcal{E}^1(X,\{\omega\})\right)$.

Let us describe the contents of this chapter. We first recall some definitions and known facts. In Section 3.2 we prove Propositions B and C and we give some concrete examples of measures with finite energy. We then discuss the invariance properties of finite energy measures and we give a counterexample insuring the non invariance under bimeromorhic maps (Section 3.3).

3.1 Preliminaries

3.1.1 Big classes and the non-pluripolar product

Let X be a compact Kähler manifold of complex dimension n and let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a real $(1,1)$-cohomology class. Recall that α is said to be *pseudo-effective* (*psef* for short) if it can be represented by a closed positive $(1,1)$-current T. Given a smooth representative θ of the class α, it follows from $\partial\bar{\partial}$-lemma that any positive $(1,1)$-current can be written as $T = \theta + dd^c\varphi$ where the global potential φ is a θ-plurisubharmonic (θ-psh for short) function, i.e. $\theta + dd^c\varphi \geq 0$. Here, d and d^c are real differential operators defined as

$$d := \partial + \bar{\partial}, \qquad d^c := \frac{i}{2\pi}\left(\bar{\partial} - \partial\right).$$

The set of all psef classes forms a closed convex cone and its interior is by definition the set of all *big* cohomology classes.
We say that the cohomology class α is *big* if it can be represented by a *Kähler current*, i.e. if there exists a positive closed $(1,1)$-current $T_+ \in \alpha$ that dominates some (small) Kähler form. By Demailly's regularization theorem [Dem92] one can assume that $T_+ := \theta + dd^c\varphi_+$ has *analytic singularities*, namely there exists $c > 0$ such that (locally on X),

$$\varphi_+ = \frac{c}{2}\log\sum_{j=1}^{N}|f_j|^2 + u,$$

where u is smooth and $f_1, \cdots, f_N$ are local holomorphic functions.

Definition 3.1.1. *If α is a big class, we define its* ample locus $\mathrm{Amp}\,(\alpha)$ *as the set of points $x \in X$ such that there exists a strictly positive current $T \in \alpha$ with analytic singularities and smooth around x.*

Note that the ample locus $\mathrm{Amp}\,(\alpha)$ is a Zariski open subset by definition, and it is nonempty since T_+ is smooth on a Zariski open subset of X.

If T and T' are two closed positive currents on X, then T is said to be *more singular* than T' if their local potentials satisfy $\varphi \leq \varphi' + O(1)$.

A positive current T is said to have *minimal singularities* (inside its cohomology class α) if it is less singular than any other positive current in α. Its θ-psh potentials φ will correspondingly be said to have minimal singularities. Note that any θ-psh function φ with minimal singularities is locally bounded on the ample locus $\mathrm{Amp}\,(\alpha)$ since it has to satisfy $\varphi_+ \leq \varphi + O(1)$. Furthermore, such θ-psh functions with minimal singularities always exist, one can consider for example

$$V_\theta := \sup\{\varphi \ \theta\text{-psh}, \varphi \leq 0 \text{ on } X\}.$$

We now introduce the *volume* of the cohomology class $\alpha \in H^{1,1}_{big}(X,\mathbb{R})$:

Definition 3.1.2. *Let $T_{\min}$ a current with minimal singularities in α and let Ω a Zariski open set on which the potentials of $T_{\min}$ are locally bounded, then*

$$\mathrm{vol}(\alpha) := \int_\Omega T^n_{\min} > 0 \tag{3.1.1}$$

is called the volume of α.

Note that the Monge-Ampère measure of $T_{\min}$ is well defined in Ω by [BT82] and that the volume is independent of the choice of $T_{\min}$ and Ω ([BEGZ10, Theorem 1.16]). Given $T_1, ..., T_p$ closed positive $(1,1)$-currents, it has been shown in [BEGZ10] that the (multilinear) *non-pluripolar product*

$$\langle T_1 \wedge ... \wedge T_p\rangle$$

is a well defined closed positive (p,p)-current that does not charge pluripolar sets. In particular, given $\varphi_1, ..., \varphi_n$ θ-psh functions, we define their Monge-Ampère measure as

$$\mathrm{MA}\,(\varphi_1, ...\varphi_n) := \langle(\theta + dd^c\varphi_1) \wedge ... \wedge (\theta + dd^c\varphi_n)\rangle.$$

By construction the latter is a non-pluripolar measure and satisfies

$$\int_X MA(\varphi_1, ...\varphi_n) \leq \mathrm{vol}(\{\theta\}).$$

In the case $\varphi_1 = ... = \varphi_n = \varphi$ we simply set

$$\mathrm{MA}\,(\varphi) = MA(\varphi, ...\varphi).$$

By defintion of the volume of $\{\theta\}$ and the fact that the non-pluripolar product does not charge pluripolar sets, it is then clear that for any $T_{\min} = \theta + dd^c\varphi_{\min} \in \{\theta\}$ current with minimal singularities, one has

$$\int_X \mathrm{MA}\,(\varphi_{\min}) = \int_X \langle T^n_{\min}\rangle = \mathrm{vol}(\{\theta\}).$$

3.1.2 Finite (weighted) energy classes

Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class and $\theta \in \alpha$ be a smooth representative.

Definition 3.1.3. *A closed positive* $(1,1)$*-current* T *on* X *with cohomology class* α *is said to have* full Monge-Ampère mass *if*

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha).$$

We denote by $\mathcal{E}(X, \alpha)$ *the set of such currents. If* φ *is a* θ*-psh function such that* $T = \theta + dd^c\varphi$*, we will say that* φ *has* full Monge-Ampère mass *if* $\theta + dd^c\varphi$ *has full Monge-Ampère mass. We denote by* $\mathcal{E}(X, \theta)$ *the set of corresponding functions.*

Currents with full Monge-Ampère mass have mild singularities, in particular they have zero Lelong number at every point $x \in \mathrm{Amp}\,(\alpha)$ (see [DN13, Proposition 1.9]).

Definition 3.1.4. *We define the energy of a* θ*-psh function* φ *as*

$$E_\theta(\varphi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X -(\varphi - V_\theta)\langle T^j \wedge \theta_{\min}^{n-j} \rangle \ \in]-\infty, +\infty]$$

with $T = \theta + dd^c\varphi$ *and* $\theta_{\min} = \theta + dd^c V_\theta$*. We set*

$$\mathcal{E}^1(X, \theta) := \{\varphi \in \mathcal{E}(X, \theta) \mid E_\theta(\varphi) < +\infty\}.$$

We denote by $\mathcal{E}^1(X, \alpha)$ *the set of positive currents in the class* α *whose global potential has finite energy.*

The energy functional is non-increasing and for an arbitrary θ-psh function φ,

$$E_\theta(\varphi) := \sup_{\psi \geq \varphi} E_\theta(\psi) \in]-\infty, +\infty]$$

over all $\psi \geq \varphi$ with minimal singularities (see [BEGZ10, Proposition 2.8]).

3.2 Finite Energy Measures

Let X be a compact Kähler manifold of complex dimension n and α be a big class and $\theta \in \alpha$ be a smooth representative. The following notion has been introduced in [BBGZ13]:

Definition 3.2.1. A probability measure μ on X has finite energy in α iff there exists $T \in \mathcal{E}^1(X, \alpha)$ such that

$$\mu = \frac{\langle T^n \rangle}{\mathrm{vol}(\alpha)}. \tag{3.2.1}$$

In this case we write $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha))$.

The purpose of this note is to study the set $\mathrm{MA}\,(\mathcal{E}^1(X,\alpha))$ of finite energy measures.

3.2.1 Some Criteria

Let us recall that a probability measure μ having finite energy is necessarily non-pluripolar (see [BBGZ13, Lemma 4.4]).

When (X,ω) is a compact Riemann surface ($n=1$) then $\mu=\omega+dd^c\varphi\in \mathrm{MA}\,(\mathcal{E}^1(X,\{\omega\}))$ iff φ belongs to the Sobolev space $W^{1,2}(X)$. This follows from Stokes theorem since

$$\int_X(-\varphi)d\mu=\int_X(-\varphi)\omega+\int_X d\varphi\wedge d^c\varphi.$$

We recall that a probability measure μ has finite measure iff for any $\psi\in \mathcal{E}^1(X,\theta)$

$$\int_X-(\psi-V_\theta)d\mu<+\infty,$$

where V_θ is the θ-psh function with minimal singularities defined in Section 3.1 (see [BBGZ13, Lemma 4.4]). In particular, this insures that the set of measures with finite energy in a given cohomology class is convex, since given $\mu_1,\mu_2\in \mathrm{MA}\,(\mathcal{E}^1(X,\{\theta\}))$, then clearly for any $t\in[0,1]$,

$$\int_X-(\psi-V_\theta)\ (td\mu_1+(1-t)d\mu_2)<+\infty.$$

Let μ,ν be two probability measures such that $\mu\lesssim\nu$. An immediate consequence of the above characterization is that μ has finite energy in α if so does ν. We now give a techinical criteria to insure that a given probability measure has finite energy.

Lemma 3.2.2. *Assume $\omega\in\alpha$ is a Kähler form. Let $\psi\in\mathcal{E}^1(X,\omega/2)$. Assume there exists a constant $A>0$ such that*

$$\mu\le A\,\mathrm{Cap}_\psi^{1+\varepsilon}$$

for some $\varepsilon>0$. Then μ has finite energy in α.

By Cap_ψ we mean here the generalized Monge-Ampère capacity introduced and studied in [DNL14a, DNL14b], namely for any Borel set $E\subset X$,

$$\mathrm{Cap}_\psi(E):=\sup\left\{\int_E \mathrm{MA}\,(u)\quad\mid\quad u\in \mathrm{PSH}(X,\omega),\ \psi-1\le u\le\psi\right\}.$$

Proof. We will follow the arguments in [DNL14a, Theorem 3.1]. We normalize ω such that $\int_X\omega^n=1$ and recall that given a probability measure μ, there exists a unique (up to constant) $\varphi\in\mathcal{E}(X,\omega)$ such that

$$\mu=(\omega+dd^c\varphi)^n.$$

Set

$$H(t) = \left[\mathrm{Cap}_\psi(\{\varphi < \psi - t\})\right]^{1/n}, \ t > 0.$$

Observe that $H(t)$ is right-continuous and $H(+\infty) = 0$ (see [DNL14a, Lemma 2.6]). It follows from [DNL14a, Lemma 2.7] that $\mathrm{Cap}_\omega \leq 2^n \mathrm{Cap}_\psi$. Using [DNL14a, Proposition 2.8] and the assumption on the measure $\mathrm{MA}\,(\varphi)$, we get

$$\begin{aligned} s^n \mathrm{Cap}_\psi(\{\varphi < \psi - t - s\}) &\leq \int_{\{\varphi < \psi - t\}} \mathrm{MA}\,(\varphi) \\ &\leq A\left[\mathrm{Cap}_\psi(\{\varphi < \psi - t\})\right]^{1+\varepsilon}, \end{aligned}$$

We then get

$$sH(t+s) \leq A^{1/n} H(t)^{1+\varepsilon}, \ \forall t > 0, \forall s \in [0,1].$$

Then by [EGZ09, Lemma 2.4] we get $\varphi \geq \psi - C$, where C only depends on A. This implies $\varphi \in \mathcal{E}^1(X,\omega)$ and so $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\{\omega\}))$. □

We stress that the above result still holds when $\omega \geq 0$ is merely semipositive.

3.2.2 Measures with densities

Let α be a Kähler class and $\omega \in \alpha$ be a Kähler form. We consider probability measures of the type $\mu = f\omega^n$ where the density $0 < f \in L^1(X)$. We investigate under which assumptions on the density f, the measure μ has finite energy. We recall that by [GZ07] there exists a unique (up to constant) ω-psh function $\varphi \in \mathcal{E}(X,\omega)$ solving

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{3.2.2}$$

When $f \in L^p(X)$ for some $p > 1$, it follows from the work of Kołodziej [Koł98] that the solution of (3.2.2) is actually uniformly bounded (and even Hölder continuous) on the whole of X. In particular, $\varphi \in \mathcal{E}^1(X,\omega)$ that means $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\{\omega\}))$. In the following we consider concrete cases when the density f is merely in $L^1(X)$.

If the density has *finite entropy*, i.e. $\int_X f \log f < +\infty$, then the measure has finite energy (see [BBEG11, Lemma 2.18]). As we will see in the sequel this condition is very strong and it is not necessary.

3.2.3 Radial measures

We consider here radially invariant measures. For simplicity we work in the local case but the same type of computations can be done in the compact setting. Let $\chi : \mathbb{R} \to \mathbb{R}$ a convex increasing function such that $\chi'(-\infty) = 0$ and $\chi(t) = t$ for $t > 0$. Denote by $\|z\| = \sqrt{|z_1|^2 + ... + |z_2|^2}$ the Euclidean norm of $\mathbb{C}^n$. Consider

$$\varphi(z) = \chi \circ \log \|z\|.$$

Then φ is plurisubharmonic in $\mathbb{B}(0, r) \subset \mathbb{C}^n$ with $r > 0$ small, and

$$\mu := (dd^c\varphi)^n.$$

Observe that, giving a radial measure in $\mathbb{B}(0, r)$ is the same thing as giving a positive measure ν in the interval $(0, r]$. This means that μ has fintie energy if and only if

$$\int_0^r |\chi(\log \rho)| d\nu(\rho) < \infty.$$

Smooth weights

Assume now that $\chi : \mathbb{R} \to \mathbb{R}$ is C^2. Then, by a simple computation we get

$$\mu = (dd^c\varphi)^n = f dV, \quad \text{with} \quad f(z) = \frac{c_n(\chi' \circ \log \|z\|)^{n-1}\chi''(\log \|z\|)}{\|z\|^{2n}}$$

where dV denotes the Euclidean measure on $\mathbb{C}^n$. It turns out that μ has finite energy iff

$$\int_{\mathbb{B}(0,r)} -\chi \circ \log \|z\| f(z) dV < +\infty,$$

that, using polar coordinates, is equivalent to

$$\int_{-\infty}^{\log r} -\chi(s)(\chi'(s))^{n-1}\chi''(s) ds < +\infty. \tag{3.2.3}$$

Example. Consider $\chi_p(t) = -(-t)^p$ with $0 < p < 1$. Then the associated radial measure μ_p has finite energie iff $p < \frac{n}{n+1}$.

In [DDG$^+$, Corollary 4.4], the authors have proven that the range of $\mathrm{MAH}\,(X, \omega)$, the Monge Ampère operator of plurisubsharmonic Hölder continuous functions, has the L^p property: if $\mu \in \mathrm{MAH}\,(X, \omega)$ and $0 \le g \in L^p(\mu)$ for some $p > 1$ with $\int_X g d\mu = \int_X \omega^n$, then $g\mu \in \mathrm{MAH}\,(X, \omega)$. One can wonder whether $\mathrm{MA}\,(\mathcal{E}^1(X, \omega))$, i.e. the set of finite energy measures, satisfies such a property. This is not the case as the following example shows.
Let $n > 1$ and $\mu = f\omega^n = (\omega + dd^c\chi \circ \log \|z\|)^n$ where $\chi(t) := -(-t)^{\frac{n-1}{n+1}}$. Then $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \omega))$. We now consider $g(z) = (-\log \|z\|)^{n/(n+1)}$ and observe that $g \in L^{\frac{n+1}{n}}(\mu)$. But then $g\mu \notin \mathrm{MA}\,(\mathcal{E}^1(X, \omega))$ since one can check that

$$g\mu \sim (\omega + dd^c\chi_1 \circ \log \|z\|)^n,$$

where $\chi_1(t) = -(-t)^{n/(n+1)}$ and then the integral in (3.2.3) is not finite.

Remark 3.2.3. Consider $H \subset X$ a smooth real hypersurface and set μ_H the Lebesgue measure on H. Then it follows from [Zer04, Theorem 5.1] that for any Borel set $E \subset X$

$$\mu_H(E) \leq \mathrm{Cap}_\omega(E)^2,$$

where Cap_ω is the classical Monge-Ampère capacity defined as

$$\mathrm{Cap}_\omega(E) := \sup\left\{ \int_E (\omega + dd^c u)^n \quad | \quad u \in \mathrm{PSH}(X,\omega) \;\; -1 \leq u \leq 0 \right\}.$$

Using Kołodziej's approach [Koł98], we then get that $\mu_H \in MAH(X,\omega)$. In particular, μ_H has finite energy.

A concrete example is $H = \mathbb{S}^{2n-1} \subset \mathbb{C}^n$. In this case $\mu_{\mathbb{S}^{2n-1}} = \mathrm{MA}\,(\varphi)$ where $\varphi \sim \log^+ \|z\|$ and $\nu(\rho) = \delta_1$.

3.2.4 Divisorial singularities

Let $D = \sum_{j=1}^N D_j$ be a simple normal crossing divisor on X. Here "simple normal crossing" means that around each intersection point of k components $D_{j_1}, ..., D_{j_k}$ $(k \leq N)$, we can find complex coordinates $z_1, ..., z_n$ such that for each $l = 1, ..., k$ the hypersurface D_{j_l} is locally given by $z_l = 0$. For each j, let L_j be the holomorphic line bundle defined by D_j. Let s_j be a holomorphic section of L_j defining D_j, i.e $D_j = \{s_j = 0\}$. We fix a hermitian metric h_j on L_j such that $|s_j| := |s_j|_{h_j} \leq 1/e$. We say that f satisfies Condition $C(B,\alpha)$ for some $B > 0$, $\alpha > 0$ if

$$f = \frac{h}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^{1+\alpha}}. \tag{3.2.4}$$

where $h \in C^\infty(X)$, $1/B \leq h \leq B$.

Proposition 3.2.4. *Assume that f satisfies $C(B,\alpha)$ for some $B > 0$, $\alpha > 0$. Then the following holds:*

1. *If $\alpha > 1/2$, then $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \{\omega\}))$.*
2. *If $\alpha \leq 1/2$, then $\mu \notin \mathrm{MA}\,(\mathcal{E}^1(X, \{\omega\}))$.*

Proof. When $\alpha > 1/2$, by [DNL14a, Theorem 2] we can find $q \in (1-\alpha, 1/2)$ such that

$$\sum_{j=1}^N -a_1(-\log|s_j|)^q - A_1 \leq \varphi,$$

where $a_1, A_1 > 0$ depends on B, α, q.
Note that the function $u_p = \sum_{j=1}^N -a_1(-\log|s_j|)^q$ if ω-psh is $a_1 > 0$ is small enough and that $u_q \in \mathcal{E}^1(X, \{\omega\})$, hence so does φ.

In the case $\alpha \in (0,1)$, by [DNL14a, Proposition 4.4] we get that for each $0 < p < 1-\alpha$ we have

$$\varphi \leq \sum_{j=1}^{N} -a_2(-\log|s_j|)^p + A_2,$$

where $a_2, A_2 > 0$ depend on B, α, p. Denote $u_p = \sum_{j=1}^{N} -a_2(-\log|s_j|)^p$. Observe that if $\alpha < 1/2$, we can choose $p \in (1/2, 1-\alpha)$ such that $u_p \notin \mathcal{E}^1(X,\omega)$. Thus $\varphi \notin \mathcal{E}^1(X,\omega)$ and hence the conclusion. What is missing is the case $\alpha = 1/2$. Consider $u = \sum_{j=1}^{N} -b(-\log|s_j|)^{1/2}$, where b is a small constant such that $u \in \mathrm{PSH}(X,\omega)$. Then $u \notin \mathcal{E}^1(X,\omega)$ and we can find a constant $C > 0$ such that

$$\mathrm{MA}\,(u) \leq \frac{C}{B\prod_{j=1}^{N}|s_j|^2(-\log|s_j|)^{3/2}},$$

hence the conclusion. □

Remark 3.2.5. *Observe that in this case the* entropy condition, $\int_X f\log f < +\infty$, *is satisfied only for* $\alpha > 1$ *although the measure has finite measure as soon as* $\alpha > 1/2$.

3.3 (Non) Stability of Finite Energy Measures

A natural question that comes up is about stability of measures having finite enegy. More precisely, given X, Y compact Kähler manifolds of complex dimension n, m, respectively, with $m \leq n$ and $f : X \to Y$ a holomorphic map, one can study the stability properties of finite energy measures under f.
It turns out that finite energy measures are invariant under biholomorphism but not under bimeromorphism as we explain in Sections 3.3.1 and 3.3.2.

In the following we wonder whether this notion depends or not on the cohomology class. In other words, given α, β big classes and a probability measure $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\alpha))$, we ask whether $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X,\beta))$ or not.
We recall that by [BEGZ10, Theorem 3.1], there exists a unique positive current $S \in \mathcal{E}(X,\beta)$ such that

$$\mu = \frac{\langle S^n \rangle}{\mathrm{vol}(\beta)}.$$

Therefore the question reduces asking if $S \in \mathcal{E}^1(X,\beta)$ or not. It turns out that this is false in general (see Counterexample 3.3.2). We obtain a positive answer under restrictive conditions on the cohomology classes, i.e. α, β both Kähler, as Proposition 3.3.1 shows.

3.3.1 Invariance property

Finite energy measures are invariant under biholomorphism. Indeed, if $f : X \to Y$ is a biholomorphic map (in particular $n = m$) and $\alpha \in H^{1,1}_{big}(X, \mathbb{R})$ then

$$\mu \in \mathcal{E}^1(X, \alpha) \quad \text{if and only if} \quad f_\star\mu \in \mathcal{E}^1(Y, f_\star\alpha).$$

This is a consequence of the fact that if we write $\mu = \langle T^n \rangle$ then $f_\star\mu = \langle (f_\star T)^n \rangle$ and

$$T \in \mathcal{E}^1(X, \alpha) \Longleftrightarrow f_\star T \in \mathcal{E}^1(Y, f_\star\alpha),$$

(see [DN13]).

Proposition 3.3.1. *Let α, β be Kähler classes and μ a probability measure. Then*

$$\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha)) \Longleftrightarrow \mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \beta)).$$

Proof. Pick ω_1 and ω_2 Kähler forms as smooth representiatives of α and β, respectively. We suppose $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha))$ and we write

$$\mu = \frac{(\omega_1 + dd^c\varphi_\mu)^n}{\mathrm{vol}(\alpha)}.$$

We want to show that there exists $\psi_\mu \in \mathcal{E}^1(X, \omega_2)$ such that $\mu = \frac{(\omega_2 + dd^c\psi_\mu)^n}{\mathrm{vol}(\beta)}$. By [GZ07, Theorem 4.2], it is equivalent to showing that $\mathcal{E}^1(X, \omega_2) \subset L^1(\mu)$. We recall that since ω_1, ω_2 are Kähler forms, there exists $C > 1$ such that $\omega_1 \leq C\omega_2$. Now, for all $\psi \in \mathcal{E}^1(X, \omega_2)$, $\psi \leq 0$,

$$\begin{aligned} \int_X (-\psi) d\mu &= \frac{1}{\mathrm{vol}(\alpha)} \int_X (-\psi)(\omega_1 + dd^c\varphi_\mu)^n \\ &\leq \frac{1}{\mathrm{vol}(\alpha)} \int_X (-\psi)(C\omega_2 + dd^c\varphi_\mu)^n < +\infty. \end{aligned}$$

The finitess of the above integral follows from [GZ07, Proposition 2.5] and from the fact that [DN13, Theorem 3.1] insures $\psi, \varphi_\mu \in \mathcal{E}^1(X, C\omega_2)$. □

3.3.2 Non invariance property

The notion of finite energy for non pluripolar measures is not invariant under bimeromorphic changing of coordinates. Indeed, the Example below points out that $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \{\tilde{\omega}\}))$ but $\pi_\star\mu \notin \mathrm{MA}\,(\mathcal{E}^1(\mathbb{P}^2, \{\lambda\omega_{FS}\}))$ for any $\lambda > 0$.

More generally, Definition 3.2.1 depends on the cohomology class: in the following we show that given α, β big classes, there exists a measure μ such that $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha))$ but $\mu \notin \mathrm{MA}\,(\mathcal{E}^1(X, \beta))$.

Example 3.3.2. Let $\pi : X \to \mathbb{P}^2$ be the blow up at one point p and set $E := \pi^{-1}(p)$. Let U be a local chart of $\mathbb{P}^2$ such that $p \to (0,0) \in U$. Fix a positive $(1,1)$-current ω' on $\mathbb{P}^2$ such that its global potential on U can be written as $\epsilon\,\eta(z)\log\|z\|$ where η is a cut-off fucntion so that $\eta \equiv 1$ on $\mathbb{B}$, $\eta \equiv 0$ on $U \setminus \mathbb{B}(2)$ and $\epsilon > 0$ is small enough. Then $\tilde{\omega} := (\pi^\star\omega' - [E]) + \pi^\star\omega_{FS}$ is a Kähler form and clearly $\tilde{\omega} \geq \pi^\star\omega_{FS}$. Let $\alpha = \{\tilde{\omega}\}$ and $\beta = \pi^\star\{\omega_{FS}\}$ with $\mathrm{vol}(\omega_{FS}) = 1$. On U we define

$$\varphi_p := \frac{1}{C}\eta \cdot u_p - K_p$$

where $u_p := -(-\log\|z\|)^p$, K_p is a positive constant such that $\varphi_p \leq -1$ and $C > 0$. Choosing C big enough φ_p induces a ω_{FS}-psh function on $\mathbb{P}^2$, say $\tilde{\varphi}_p$. For $p = \frac{1}{2} - \delta$ with $\delta > 0$ small enough, we set

$$\mu := \frac{(\tilde{\omega} + dd^c\pi^\star\tilde{\varphi}_p)^2}{\mathrm{vol}(\tilde{\omega})}.$$

We will show that $\mu \notin \mathrm{MA}\,(\mathcal{E}^1(X,\beta))$, or better that there exists a function $\psi \in \mathcal{E}^1(X, \pi^\star\omega_{FS})$, $\psi \leq 0$, such that $\int_X(-\psi)d\mu = +\infty$ (see [GZ07, Theorem 4.2]). We pick $\psi := \pi^\star\tilde{\varphi}_\varepsilon$ with $\varepsilon = \frac{2}{3} - \delta'$, $\delta' > 0$ small enough. Observe that $\psi \in \mathcal{E}^1(X, \pi^\star\omega_{FS})$ but $\psi \notin \mathcal{E}^1(X, \tilde{\omega})$ (see [DN13, Example 3.5]). We claim that $\int_X(-\pi^\star\tilde{\varphi}_\varepsilon)(\tilde{\omega} + dd^c\pi^\star\tilde{\varphi}_p)^2 = +\infty$. First note that on $\mathbb{P}^2 \setminus \{p\}$,

$$\begin{aligned} \mathrm{vol}(\tilde{\omega})\pi_\star\mu &= (\omega' + \omega_{FS} + dd^c\tilde{\varphi}_p)^2 \\ &\geq -C'\omega_{FS}^2 + 2\omega' \wedge (\omega_{FS} + dd^c\tilde{\varphi}_p) + (\omega_{FS} + dd^c\tilde{\varphi}_p)^2. \end{aligned}$$

Thus

$$3\int_X(-\pi^\star\tilde{\varphi}_\varepsilon)d\mu = 3\int_{\mathbb{P}^2}(-\tilde{\varphi}_\varepsilon)d\pi_\star\mu \geq \tag{3.3.1}$$

$$-C'\int_{\mathbb{P}^2}(-\tilde{\varphi}_\varepsilon)\omega_{FS}^2 + 2\int_{\mathbb{P}^2}(-\tilde{\varphi}_\varepsilon)\omega' \wedge (\omega_{FS} + dd^c\tilde{\varphi}_p) + \int_{\mathbb{P}^2}(-\tilde{\varphi}_\varepsilon)(\omega_{FS} + dd^c\tilde{\varphi}_p)^2.$$

We infer that

$$\int_{\mathbb{B}(\frac{1}{2})\setminus\{(0,0)\}} |(-\log\|z\|)^\varepsilon|\, dd^c\log\|z\| \wedge dd^c[\chi(\log\|z\|)] = +\infty \tag{3.3.2}$$

where $\chi(t) = -(-t)^p$, hence the conclusion. Indeed on $\mathbb{B}(\frac{1}{2}) \setminus \{(0,0)\}$,

$$dd^c\log\|z\| \wedge dd^c[\chi(\log\|z\|)] = \frac{A}{\|z\|^4}\chi''(\log\|z\|)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$

where A is postive constant. Therefore we have

$$\begin{aligned} &\int_{\mathbb{B}(\frac{1}{2})\setminus\{(0,0)\}} \frac{1}{\|z\|^4|\log\|z\||^{2-p-\varepsilon}}dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ &= C'\int_0^{\frac{1}{2}} \frac{1}{\rho(-\log\rho)^{2-p-\varepsilon}}\,d\rho \\ &= C'\int_{-\log\frac{1}{2}}^{+\infty} \frac{1}{s^{2-p-\varepsilon}}\,ds = +\infty \end{aligned}$$

since $2 - p - \varepsilon \leq 1$.
A similar computation (by replacing ε by p) show that $\pi^\star \tilde{\varphi}_p \in \mathcal{E}^1(X, \tilde{\omega})$ and so $\mu \in \mathrm{MA}\,(\mathcal{E}^1(X, \alpha))$ by construction. This proves (*i*) of Proposition B. Note that (*ii*) follows form the computations in (3.3.1) and (3.3.2).

Chapter 4

Monge-Ampère equations on quasi-projective varieties

Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n and let D be a divisor on X. Let f be a non-negative function such that $\int_X f\omega^n = \int_X \omega^n$. Consider the following complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{4.0.1}$$

When f is smooth and positive on X, it follows from the seminal work of Yau [Yau78] that (4.0.1) admits a unique normalized smooth solution φ such that $\omega + dd^c\varphi$ is a Kähler form. Recall that this result solves in particular the Calabi conjecture and allows to construct Ricci flat metrics on X whenever $c_1(X) = 0$.

It is very natural to look for a similar result when f is merely smooth and positive on the complement of D, e.g. when studying Calabi's conjecture on quasi-projective manifolds (see e.g. [TY, TY90, TY91] and [Hei12]) for recent developments). The study of conical Kähler-Einstein metrics (Kähler-Einstein metrics in the complement of a divisor with a precise behavior near D) has played a major role in the resolution of the Yau-Tian-Donaldson conjecture for Fano manifolds (see [Don12],[DS12],[CDS12a, CDS12b, CDS13],[Tia12]).

However no systematic study of the regularity of solutions to such complex Monge-Ampère equations has ever been done, this is the main goal of this article. It follows from [GZ07] that (4.0.1) has a solution in the finite energy class $\mathcal{E}(X, \omega)$ which turns out to be the unique one up to an additive constant (see [Din09]). We say that the solution is normalized if $\sup_X \varphi = 0$. The problem thus boils down to showing that such a normalized solution is smooth in $X \setminus D$ and understanding its asymptotic behavior along D.

As in the classical case of Yau [Yau78] the main difficulty is in establishing a priori $\mathcal{C}^0$ bounds. Since, in general the solution φ is unbounded, the idea is to bound φ from below by some (singular) ω-psh function.

Our first main result shows that the solution φ is smooth in $X \setminus D$ when f satisfies the mild condition $\mathcal{H}_f$:

$$f = e^{\psi^+ - \psi^-},\ \psi^\pm \text{ are quasi plurisubharmonic on } X, \quad \psi^- \in L^\infty_{\text{loc}}(X \setminus D).$$

Let us stress that D is here an arbitrary divisor.

Theorem 1. *Assume that* $0 < f \in \mathcal{C}^\infty(X \setminus D)$ *satisfies Condition* $\mathcal{H}_f$*. Then the solution* φ *is also smooth on* $X \setminus D$*.*

The most difficult part is the C^0 estimate that relies on the following result:

Theorem 2. *Assume that* $f \le e^{-\phi}$ *for some quasi-plurisubharmonic function* ϕ*. Then for each* $a > 0$ *such that* $a\phi \in \mathrm{PSH}(X, \omega/2)$ *there exists* $A > 0$ *depending on* $\int_X e^{-2\varphi/a}\omega^n$ *such that*

$$\varphi \ge a\phi - A.$$

Remark. It follows from Skoda's theorem [Sko72] that $\int_X e^{-2\varphi/a}\omega^n$ is finite for all $a > 0$, since $\varphi \in \mathcal{E}(X, \omega)$ has zero Lelong number at all points [GZ07].

In Theorem 1, the density f is only in $L^1(X)$ and there is no regularity assumption on D. Hence we do not have any information about the behavior of φ near D. If we assume more regularity on f and D, we will get more precise $\mathcal{C}^0$-bounds.

Assume that $D = \sum_{j=1}^N D_j$ is a simple normal crossing divisor (snc for short). For each $j = 1, ..., N$, let L_j be the holomorphic line bundle defined by D_j. Let s_j be a holomorphic section of L_j such that $D_j = \{s_j = 0\}$. Fix a hermitian metric h_j on L_j such that $|s_j| := |s_j|_{h_j} \le 1/e$.

When the behavior of f near the divisor D looks exactly like

$$\frac{1}{\prod_{j=1}^N |s_j|^2 |\log |s_j||^{1+\alpha}},\ \alpha \in (0, 1]$$

we show in Proposition 4.3.4 and Proposition 4.3.5 that $\varphi(x)$ converges to $-\infty$ as x approaches D with precise rates. In particular there is no bounded solution to (4.0.1).

Theorem 3. Assume $f = \frac{h}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^{1+\alpha}}$, where $1/B \le h \le B$ on X, then the following holds:

(a) if $\alpha > 1$ then φ is continuous on X, $\varphi \ge -C$, with $C = C(B, \alpha)$.

(b) if $\alpha \in (0,1)$ then for each $0 < p < 1-\alpha$ and each $1-\alpha < q < 1$, we have

$$-a_1 \sum_{j=1}^{N} (-\log|s_j|)^q - A_1 \leq \varphi \leq -a_2 \sum_{j=1}^{N} (-\log|s_j|)^p + A_2,$$

where $a_1, A_1 > 0$ depend on B, α, q while $a_2, A_2 > 0$ depend on B, α, p.

(c) if $\alpha = 1$ and D is smooth then for any $p \in (0,1)$ there exist $a, A > 0$ depending on B, p and $A_1, A_2 > 0$ depending on B such that

$$\sum_{j=1}^{N} -A_1 \left[\log(-\log|s_j| + A_2)\right] \leq \varphi \leq -a \sum_j \left[\log(-\log|s_j|)\right]^p + A.$$

It would be interesting to obtain (c) when D is non smooth but our method only yields the weaker estimate (b) in this case.

When $f \in L^p(\omega^n)$ for some $p > 1$, it follows from the work of Kołodziej [Koł98] that the solution of (4.0.1) is actually uniformly bounded (and even Hölder continuous) on the whole of X.

In our result, the density f is merely in L^1. The first part of Theorem 3 says that when $\alpha > 1$ the solution is continuous on X. Kołodziej's result [Koł98, Theorem 2.5.2] also applies for $\alpha > n$ but can not be applied to a density f as above if $\alpha \leq n$.

Observe furthermore that $\alpha = 1$ is a critical exponent as is easily seen when $n = 1$. In any dimension, when f has singularities of Poincaré type,

$$\frac{1/C}{\prod_{j=1}^{N} |s_j|^2 |\log|s_j||^2} \leq f(z) \leq \frac{C}{\prod_{j=1}^{N} |s_j|^2 |\log|s_j||^2}$$

along D we show in Section 4.3.3 that the solution is locally uniformly bounded on compact subsets of $X \setminus D$ and goes to $-\infty$ along D with a certain rate. If moreover f has a "very precise" behavior near D it follows from the recent work of Auvray (see [Auv11]) that φ goes to $-\infty$ along D like $\sum_{j=1}^{N} -\log(-\log|s_j|)$. The assumptions needed in [Auv11] are very restrictive while in our result we only need a very weak condition on the density. Recall also that in [TY] the authors constructed "almost complete" Kähler Einstein metrics of negative Ricci curvature on $X \setminus D$. In this case the $\mathcal{C}^0$ estimate follows easily from the maximum principle.

In order to prove the $\mathcal{C}^0$-estimate we follow and generalize Kołodziej's approach. We introduce and study the ψ-Capacity of a Borel subset $E \subset X$,

$$\mathrm{Cap}_\psi(E) := \sup\left\{ \int_E (\omega + dd^c u)^n \;\middle|\; u \in \mathrm{PSH}(X,\omega),\ \psi - 1 \leq u \leq \psi \right\}$$

where $\psi \in \mathrm{PSH}(X,\omega)$ and here $(\omega + dd^c u)^n$ is the nonpluripolar Monge-Ampère measure of u (see Section 4.1 for the definition). When ψ is constant, $\psi \equiv C$, we recover the Monge-Ampère capacity,

$$\mathrm{Cap}_\omega = \mathrm{Cap}_C.$$

A similar notion has been studied in [CKZ05] in a local context. These generalized capacities are interesting for themselves. In this paper we only need some of their properties and refer the reader to [DNL14b] for a more systematic study.

One of the advantages of the Kołodziej's approach for the $\mathcal{C}^0$ estimates is that it also works in the case of semipositive and big classes as shown in [BGZ08], [EGZ09] and [BEGZ10]. Thus it is not surprising that our method is still valid in this situation.

Let θ be a smooth semipositive form on X such that $\int_X \theta^n > 0$. Let f be a non-negative function such that $\int_X f\omega^n = \int_X \theta^n$. Consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = f\omega^n. \tag{4.0.2}$$

It follows from [BBGZ13] that (4.0.2) admits a unique normalized solution $\varphi \in \mathcal{E}(X,\theta)$. As in the Kähler case, it is interesting to investigate the regularity properties of φ if we know that the density f is smooth, strictly positive outside a divisor D and verifies Condition $\mathcal{H}_f$. We can not expect φ to be smooth on $X \setminus D$ since θ may be zero somewhere there. Our result below shows that the solution is smooth on $X \setminus (D \cup E)$, where E is an effective simple normal crossing divisor on X such that $\{\theta\} - c_1(E)$ is ample.

Theorem 4. *Let (X,ω) be a compact Kähler manifold of complex dimension n and D be an arbitrary divisor on X. Let E be an effective snc divisor on X, and θ be a smooth semipositive form on X such that $\int_X \theta^n > 0$ and $\{\theta\} - c_1(E)$ is ample. Assume that $0 < f \in \mathcal{C}^\infty(X \setminus D)$ satisfies Condition $\mathcal{H}_f$. Let φ be the unique normalized solution to equation (4.0.2). Then φ is smooth on $X \setminus (D \cup E)$.*

Remark. The condition we impose on $\{\theta\}$ is natural in studying Kähler Einstein metrics on singular varieties (see [BG13]).

Let us say some words about the organization of the paper. In Section 4.1, we introduce the generalized ψ-Capacity, and establish their basic properties. The proof of Theorem 1 will be given in Section 4.2. We provide some volume-capacity estimates in Section 4.3.1. We then use these to prove Theorem 2 and 3 and discuss about the asymptotic behavior of solutions near the divisor in Section 4.3.2. Finally we consider the case of semipositive and big classes in Section 4.4.

4.1 Preliminaries

Let (X, ω) be a compact Kähler manifold. We first recall basic facts about finite energy classes of ω-psh functions on X. The reader can find more details about these in [GZ07].

4.1.1 Finite energy classes

Definition 4.1.1. We let $\mathrm{PSH}(X, \omega)$ denote the class of ω-plurisubharmonic functions (ω-psh for short) on X, i.e. the class of functions φ such that locally $\varphi = \rho + u$, where ρ is a local potential of ω and u is a plurisubharmonic function.

Let φ be some (unbounded) ω-psh function on X and consider $\varphi_j := \max(\varphi, -j)$ the canonical approximation by bounded ω-psh functions. It follows from [GZ07] that

$$\mathbf{1}_{\{\varphi_j > -j\}}(\omega + dd^c\varphi_j)^n$$

is a non-decreasing sequence of Borel measures. We denote by $(\omega + dd^c\varphi)^n$ (or $\mathrm{MA}\,(\varphi)$ for short if ω is fixed and no confusion can occur) this limit:

$$\mathrm{MA}\,(\varphi) = (\omega + dd^c\varphi)^n = \lim_{j\to+\infty} \mathbf{1}_{\{\varphi_j > -j\}}(\omega + dd^c\varphi_j)^n.$$

It was shown in [GZ07] that the Monge-Ampère measure $\mathrm{MA}\,(\varphi)$ puts no mass on pluripolar sets. This is the non-pluripolar part of the Monge-Ampère of φ. Note that its total mass $\mathrm{MA}\,(\varphi)(X)$ can take value in $\left[0, \int_X \omega^n\right]$.

Definition 4.1.2. We let $\mathcal{E}(X, \omega)$ denote the class of ω-psh function having full Monge-Ampère mass:

$$\mathcal{E}(X, \omega) := \left\{\varphi \in \mathrm{PSH}(X, \omega) \;\Big|\; \int_X \mathrm{MA}\,(\varphi) = \int_X \omega^n\right\}.$$

Let us stress that ω-psh functions with full Monge-Ampère mass have mild singularities. Indeed, it was shown in [GZ07, Corollary 1.8] that

$$\nu(\varphi, x) = 0, \forall \varphi \in \mathcal{E}(X, \omega),\ x \in X.$$

We also recall that, for every $\varphi \in \mathcal{E}(X, \omega)$ and $\psi \in \mathrm{PSH}(X, \omega)$, the *generalized comparison principle* holds (see [BEGZ10, Corollary 2.3]), namely

$$\int_{\{\varphi<\psi\}} (\omega + dd^c\psi)^n \le \int_{\{\varphi<\psi\}} (\omega + dd^c\varphi)^n.$$

Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function such that $\chi(0) = 0$ and $\chi(-\infty) = -\infty$.

Definition 4.1.3. Let $\mathcal{E}_\chi(X,\omega)$ denote the set of ω-psh functions with finite χ-energy,

$$\mathcal{E}_\chi(X,\omega) := \{\varphi \in \mathcal{E}(X,\omega) \quad | \quad \chi(-|\varphi|) \in L^1(\mathrm{MA}\,(\varphi))\}.$$

For $p > 0$, we use the notation

$$\mathcal{E}^p(X,\omega) := \mathcal{E}_\chi(X,\omega), \text{ when } \chi(t) = -(-t)^p.$$

4.1.2 The ψ-Capacity

Definition 4.1.4. Let $\psi \in \mathrm{PSH}(X,\omega)$. We define the ψ-Capacity of a Borel subset $E \subset X$ by

$$\mathrm{Cap}_\psi(E) := \sup\left\{\int_E \mathrm{MA}\,(u) \mid u \in \mathrm{PSH}(X,\omega),\ \psi - 1 \le u \le \psi\right\}.$$

Then the Monge-Ampère capacity corresponds to $\psi \equiv$ constant (see [BT82], [Koł03], [GZ05]). We list below some basic properties of the ψ-Capacity.

Proposition 4.1.5. *(i) If $E_1 \subset E_2 \subset X$ then* $\mathrm{Cap}_\psi(E_1) \le \mathrm{Cap}_\psi(E_2)$.

(ii) If $E_1, E_2, ...$ are Borel subsets of X then

$$\mathrm{Cap}_\psi\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_{j=1}^{+\infty} \mathrm{Cap}_\psi(E_j).$$

(iii) If $E_1 \subset E_2 \subset ...$ are Borel subsets of X then

$$\mathrm{Cap}_\psi\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to+\infty} \mathrm{Cap}_\psi(E_j).$$

The following results are elementary and important for the sequel. We stress that these results still hold in the case when ω is merely semipositive and big rather than Kähler.

Lemma 4.1.6. *Let $\psi \in \mathrm{PSH}(X,\omega)$ and $\varphi \in \mathcal{E}(X,\omega)$. Then the function*

$$H(t) := \mathrm{Cap}_\psi(\{\varphi < \psi - t\}),\ t \in \mathbb{R},$$

is right-continuous and $H(t) \to 0$ as $t \to +\infty$.

Proof. The right-continuity of H follows from (iii) of Proposition 4.1.5. Let us prove the second statement. We can assume that $\psi \le 0$ on X. Fix $v \in \mathrm{PSH}(X,\omega)$ such that $\psi - 1 \le v \le \psi$. We apply the comparison principle to obtain

$$\int_{\{\varphi<\psi-t\}} \mathrm{MA}\,(v) \le \int_{\{\varphi<v-t+1\}} \mathrm{MA}\,(v) \le \int_{\{\varphi<-t+1\}} \mathrm{MA}\,(\varphi).$$

The last term goes to zero as t goes to $+\infty$ since $\varphi \in \mathcal{E}(X,\omega)$. □

Lemma 4.1.7. *Let (X, ω) be a compact Kähler manifold and $\psi \in \mathrm{PSH}(X, \omega/2)$. Then we have*

$$\mathrm{Cap}_{\omega/2}(E) \leq \mathrm{Cap}_{\psi}(E).$$

Here, $\mathrm{Cap}_{\omega/2}$ is the Monge-Ampère Capacity with respect to the Kähler metric $\omega/2$ introduced in [Koł03] and studied in [GZ05], and Cap_{ψ} is the generalized ψ-Capacity with respect to the Kähler metric ω.

We stress that the above result insures $\mathrm{Cap}_{\psi}(E) > 0$ for any Borel subset E which is not pluripolar.

Proof. Let $u \in \mathrm{PSH}(X, \omega/2)$ be such that $-1 \leq u \leq 0$. Then $\varphi := \psi + u$ is a candidate defining Cap_{ψ}. Using the definition of the Monge-Ampère meausure it is not difficult to see that

$$\int_E (\omega/2 + dd^c u)^n \leq \int_E (\omega + dd^c \varphi)^n \leq \mathrm{Cap}_{\psi}(E),$$

and taking the supremum over all u we get the result. □

The following result generalizes Lemma 2.3 in [EGZ09].

Proposition 4.1.8. *Let $\varphi \in \mathcal{E}(X, \omega)$, $\psi \in \mathrm{PSH}(X, \omega)$. Then for all $t > 0$ and $0 \leq s \leq 1$ we have*

$$s^n \mathrm{Cap}_{\psi}(\{\varphi < \psi - t - s\}) \leq \int_{\{\varphi < \psi - t\}} \mathrm{MA}\,(\varphi).$$

Proof. Let $u \in \mathrm{PSH}(X, \omega)$ such that $\psi - 1 \leq u \leq \psi$. Observe the following trivial inclusion

$$\{\varphi < \psi - t - s\} \subset \{\varphi < su + (1-s)\psi - t\} \subset \{\varphi < \psi - t\}\,.$$

It thus follows from the *generalized comparison principle* (see [BEGZ10, Corollary 2.3]) that

$$\begin{aligned} s^n \int_{\{\varphi < \psi - t - s\}} \mathrm{MA}\,(u) &\leq \int_{\{\varphi < \psi - t - s\}} \mathrm{MA}\,(su + (1-s)\psi) \\ &\leq \int_{\{\varphi < su + (1-s)\psi - t\}} \mathrm{MA}\,(su + (1-s)\psi) \\ &\leq \int_{\{\varphi < \psi - t\}} \mathrm{MA}\,(\varphi). \end{aligned}$$

By taking the supremum over all candidates u we get the result. □

4.2 Smooth solution in a general case

In this section we prove Theorem 1. The most difficult part is the $\mathcal{C}^0$ estimate which follows from Theorem 4.2.1 below.

4.2.1 Uniform estimate

In this subsection we assume that $0 \leq f \in L^1(X)$ is such that $\int_X f\omega^n = \int_X \omega^n$. Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique normalized solution to

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{4.2.1}$$

Here we normalize φ such that $\sup_X \varphi = 0$. We prove the following $\mathcal{C}^0$ estimate:

Theorem 4.2.1. *Assume that $f \leq e^{-\phi}$ for some quasi-plurisubharmonic function ϕ. Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique normalized solution to (4.2.1). Then for any $a > 0$ such that $a\phi \in \mathrm{PSH}(X,\omega/2)$, there exists $A > 0$ depending only on $\int_X e^{-2\varphi/a}\omega^n$ such that*

$$\varphi \geq a\phi - A.$$

Moreover, if ϕ is bounded in a compact subset $K \subset X$ then φ is continuous on K.

Remark 4.2.2. We stress here that our estimate above makes sense. Indeed, it follows from [GZ07] that all functions in $\mathcal{E}(X,\omega)$ have zero Lelong number at all points of X. Then by Skoda's integrability theorem we know that $e^{-B\varphi}$ is integrable for every $B > 0$ and every $\varphi \in \mathcal{E}(X,\omega)$. We stress also that the constant in our estimate only depends on an upper bound of $\int_X e^{-2\varphi/a}$.

Proof. We can assume that $\phi \leq 0$. Fix $a > 0$ such that $\psi := a\phi$ belongs to $\mathrm{PSH}(X,\omega/2)$. It follows from Lemma 4.1.7 that $\mathrm{Cap}_\omega \leq 2^n \mathrm{Cap}_{\omega/2} \leq 2^n \mathrm{Cap}_\psi$. Fix $s \in [0,1]$, $t > 0$ and apply Proposition 4.1.8 to get

$$s^n \mathrm{Cap}_\psi(\varphi < \psi - t - s) \leq \int_{\{\varphi<\psi-t\}} \mathrm{MA}\,(\varphi). \tag{4.2.2}$$

By assumption on f we have

$$\int_{\{\varphi<\psi-t\}} \mathrm{MA}\,(\varphi) \leq \int_{\{\varphi<\psi-t\}} e^{-\varphi/a} e^{\psi/a} \mathrm{MA}\,(\varphi) \leq \int_{\{\varphi<\psi-t\}} e^{-\varphi/a}\omega^n. \tag{4.2.3}$$

It follows from [GZ05] that

$$\mathrm{vol}_\omega \leq \exp\left(\frac{-C_1}{\mathrm{Cap}_\omega^{1/n}}\right).$$

Thus using Hölder inequality we get from (4.2.2) and (4.2.3) that

$$s^n \mathrm{Cap}_\psi(\varphi < \psi - t - s) \leq C_2\left(\mathrm{Cap}_\omega(\varphi < \psi - t)\right)^2 \leq C_3\left(\mathrm{Cap}_\psi(\varphi < \psi - t)\right)^2,$$

where C_3 depends only on $\int_X e^{-2\varphi/a}\omega^n$. Now, consider the following function

$$H(t) = \left[\mathrm{Cap}_\psi(\{\varphi < \psi - t\})\right]^{1/n},\ t > 0.$$

By the arguments above we get

$$sH(t+s) \leq C_4 H(t)^2,\ \forall t > 0, \forall s \in [0,1],$$

where $C_4 > 0$ depends only on $\int_X e^{-2\varphi/a}\omega^n$. It follows from Lemma 4.1.6 that H is right-continuous and $H(+\infty) = 0$. Thus by [EGZ09, Lemma 2.4] we get $\varphi \geq \psi - C_5$, where C_5 only depends on $\int_X e^{-2\varphi/a}\omega^n$. Indeed, the constant C_5 can be made very precise as follows. It follows from [EGZ09, Lemma 2.4] that there exists t_∞ such that $H(t) = 0$ if $t \geq t_\infty$. Here, we can take

$$t_\infty = 2 + s_0,$$

where $s_0 > 0$ is big enough such that

$$H(s_0) \leq \frac{1}{2C_4}.$$

By using Hölder's inequality it follows from (4.2.2) (take $s = 1$) and (4.2.3) that

$$\begin{aligned} H(t+1)^n &\leq \left(\int_X e^{-2\varphi/a}\omega^n\right)^{1/2} \left(\int_{\{\varphi<\psi-t\}} \omega^n\right)^{1/2} \\ &\leq \left(\int_X e^{-2\varphi/a}\omega^n\right)^{1/2} \left(\frac{1}{t}\int_X (-\varphi)\omega^n\right)^{1/2}. \end{aligned}$$

The last integral is bounded by a uniform constant since φ is normalized by $\sup_X \varphi = 0$ (see [GZ05]). From this we can choose $s_0 > 0$ *depending only* on an upper bound of $\int_X e^{-2\varphi/a}\omega^n$.

Now, assume that ϕ is bounded on a compact subset $K \subset X$. Set $\psi := a\phi$ as above. Let us prove that φ is continuous on K. For convenience, we normalize φ so that $\sup_X \varphi = -1$. Let $0 \geq \varphi_j$ be a sequence of continuous ω-psh functions on X decreasing to φ. Fix $\lambda \in (0,1)$. For each $j \in \mathbb{N}$ set

$$\psi_j := \lambda\varphi_j + (1-\lambda)\psi - (1-\lambda)A - 2(1-\lambda).$$

Then ψ_j belongs to $\mathrm{PSH}(X, \frac{1+\lambda}{2}\omega)$ and $\psi_j \leq \varphi_j - 2(1-\lambda)$. Set

$$H_j(t) := \left[\mathrm{Cap}_{\psi_j}(\{\varphi < \psi_j - t\})\right]^{1/n}, t > 0.$$

We can argue as above and use Proposition 4.1.8 to get

$$sH_j(t+s) \leq C_1 H_j(t)^2,\ \forall t > 0,\ \forall s \in [0,1],$$

where $C_1 > 0$ depends on $\int_X e^{-2\varphi/(1-\lambda)a}$. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing convex weight such that $\chi(0) = 0$, $\chi(-\infty) = -\infty$ and $\varphi \in \mathcal{E}_\chi(X, \omega)$. By the comparison principle we also get

$$\begin{aligned}(1-\lambda)^n \mathrm{Cap}_{\psi_j}(\varphi < \psi_j) &\leq \int_{\{\varphi < \psi_j + 1 - \lambda\}} \mathrm{MA}\,(\varphi) \leq \int_{\{\varphi < \varphi_j - (1-\lambda)\}} \mathrm{MA}\,(\varphi) \\ &\leq \frac{1}{-\chi(-1+\lambda)} \int_X (-\chi \circ (\varphi - \varphi_j)) f \omega^n.\end{aligned}$$

The latter converges to 0 as $j \to +\infty$, since φ_j decreases to φ. Thus for j big enough we have $H_j(0) \leq 1/(2C_1)$. It then follows from [EGZ09, Remark 2.5] that $H_j(t) = 0$ if $t \geq t_\infty$ where $t_\infty \leq C_2 H_j(0)$ and C_2 depends on C_1. We then get

$$\varphi \geq \lambda \varphi_j + (1-\lambda)\psi - (1-\lambda)(A+2) - C_2 H_j(0).$$

Now, letting $j \to +\infty$, we get

$$\lim_{j \to +\infty} \inf_K (\varphi - \varphi_j) \geq (\lambda - 1)(\sup_K |\psi| + A + 2).$$

Finally, letting $\lambda \to 1$ we get the continuity of φ on K. □

4.2.2 Laplacian estimate

The following a priori estimate generalizes [Pău08].

Theorem 4.2.3. *Let μ be a positive measure on X of the form $\mu = e^{\psi^+ - \psi^-}\omega^n$ where ψ^+, ψ^- are smooth on X. Let $\varphi \in \mathcal{C}^\infty(X)$ be such that $\sup_X \varphi = 0$ and*

$$(\omega + dd^c\varphi)^n = e^{\psi^+ - \psi^-}\omega^n.$$

Assume given a constant $C > 0$ such that

$$dd^c\psi^\pm \geq -C\omega, \quad \sup_X \psi^+ \leq C.$$

Assume also that the holomorphic bisectional curvature of ω is bounded from below by $-C$. Then there exists $A > 0$ depending on C and $\int_X e^{-2(4C+1)\varphi}\omega^n$ such that

$$0 \leq n + \Delta_\omega \varphi \leq A e^{-2\psi^-}.$$

We follow the lines in Appendix B of [BBEG11]. We recall the following result:

Lemma 4.2.4. *Let α, β be positive $(1,1)$-forms. Then*

$$n\left(\frac{\alpha^n}{\beta^n}\right)^{\frac{1}{n}} \leq \mathrm{tr}_\beta(\alpha) \leq n\left(\frac{\alpha^n}{\beta^n}\right) \cdot (\mathrm{tr}_\alpha(\beta))^{n-1}.$$

Proof of Theorem 4.2.3. Set $\omega_\varphi := \omega + dd^c\varphi$. Since the holomorphic bisectional curvature of ω is bounded from below by $-C$, it follows from Lemma 2.2 in [CGP11] that

$$\Delta_{\omega_\varphi} \log \mathrm{tr}_\omega(\omega_\varphi) \geq \frac{\mathrm{tr}_\omega(dd^c\psi^+ - dd^c\psi^-)}{\mathrm{tr}_\omega(\omega_\varphi)} - C\mathrm{tr}_{\omega_\varphi}(\omega). \tag{4.2.4}$$

Since $dd^c\psi^+ \geq -C\omega$, using the trivial inequality $n \leq \mathrm{tr}_\omega(\omega_\varphi)\mathrm{tr}_{\omega_\varphi}(\omega)$ we thus get from (4.2.4) that

$$\begin{aligned} \Delta_{\omega_\varphi} \log \mathrm{tr}_\omega(\omega_\varphi) &\geq -\frac{\mathrm{tr}_\omega(C\omega + dd^c\psi^-)}{\mathrm{tr}_\omega(\omega_\varphi)} - C\mathrm{tr}_{\omega_\varphi}(\omega) \\ &\geq -2C\mathrm{tr}_{\omega_\varphi}(\omega) - \frac{\Delta\psi^-}{\mathrm{tr}_\omega(\omega_\varphi)}\,. \end{aligned} \tag{4.2.5}$$

By assumption we have $0 \leq C\omega + dd^c\psi^- \leq \mathrm{tr}_{\omega_\varphi}(C\omega + dd^c\psi^-)\omega_\varphi$. Applying tr_ω to the previous inequality yields

$$Cn + \Delta\psi^- \leq (C\mathrm{tr}_{\omega_\varphi}(\omega) + \Delta_{\omega_\varphi}\psi^-)\mathrm{tr}_\omega(\omega_\varphi),$$

and hence

$$-\Delta\psi^- \geq -(C\mathrm{tr}_{\omega_\varphi}(\omega) + \Delta_{\omega_\varphi}\psi^-)\mathrm{tr}_\omega(\omega_\varphi).$$

Thus, plugging this into (4.2.5) we obtain

$$\Delta_{\omega_\varphi} \log \mathrm{tr}_\omega(\omega_\varphi) \geq -3C\mathrm{tr}_{\omega_\varphi}(\omega) - \Delta_{\omega_\varphi}\psi^-. \tag{4.2.6}$$

We want now to apply the maximum principle to the function

$$H := \log \mathrm{tr}_\omega(\omega_\varphi) + 2\psi^- - (1+4C)\varphi,$$

Let $x_0 \in X$ be such that H achieves its maximum on X at x_0. Then at x_0 we get

$$0 \geq \Delta_{\omega_\varphi} H \quad \geq \quad \mathrm{tr}_{\omega_\varphi}(\omega) - n(1+4C).$$

Furthermore, by Lemma 4.2.4 we get

$$\mathrm{tr}_\omega(\omega_\varphi)(x_0) \leq ne^{\psi^+-\psi^-}(x_0)\left(\mathrm{tr}_{\omega_\varphi}(\omega)\right)^{n-1}(x_0) \leq A_1 e^{\psi^+-\psi^-}(x_0),$$

and hence, since $\sup_X \psi^+ \leq C$,

$$\log \mathrm{tr}_\omega(\omega_\varphi)(x_0) \leq \log A_1 + \psi^+(x_0) - \psi^-(x_0) \leq A_2 - \psi^-(x_0)\,.$$

It follows that

$$H(x) \leq H(x_0) \leq A_3 + \psi^-(x_0) - (1+4C)\varphi(x_0).$$

By assumption and the $\mathcal{C}^0$ estimate in Theorem 4.2.1 we have $\varphi \geq a\psi^- - A_4$, where $a = 1/(4C+1)$ and A_4 depends on C and $\int_X e^{-2\varphi/a}\omega^n$. Thus

$$\log \operatorname{tr}_\omega(\omega_\varphi) \leq A_5 - 2\psi^-.$$

We finally infer as desired

$$\operatorname{tr}_\omega(\omega_\varphi) \leq A_6 e^{-2\psi^-}.$$

□

We are now ready to prove Theorem 1.

4.2.3 Proof of Theorem 1

Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique normalized solution to

$$(\omega + dd^c\varphi)^n = f\omega^n.$$

By assumption we can write $\log f = \psi^+ - \psi^-$, where $\psi^\pm$ are quasi psh functions on X, ψ^- is locally bounded on $X \setminus D$, and there is a uniform constant $C > 0$ such that

$$dd^c\psi^\pm \geq -C\omega, \ \sup_X \psi^+ \leq C.$$

We now approximate $\psi^\pm$ by using Demailly's regularization operator ρ_ε. We recall the construction: if u is a quasi-psh function on X and $\varepsilon > 0$ we set

$$\rho_\varepsilon(u)(z) := \frac{1}{\varepsilon^{2n}} \int_{\zeta \in T_{X,z}} u(\operatorname{exph}_z(\zeta))\chi\left(|\zeta|^2/\varepsilon^2\right) d\lambda(\zeta).$$

Here $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is a cut-off function supported in $[-1,1]$, $\int_\mathbb{R} \chi(t)dt = 1$, and

$$\operatorname{exph} : TX \to X, \quad \zeta \mapsto \operatorname{exph}_z(\zeta)$$

is the formal holomorphic part of the Taylor expansion of the exponential map defined by the metric ω. For more details, see [Dem92]. Observe that by Jensen's inequality, $\rho_\varepsilon(e^u) \geq e^{\rho_\varepsilon(u)}$. Applying this smoothing regularization to $\psi^\pm$ we get, for $\varepsilon > 0$ small enough,

$$dd^c\rho_\varepsilon(\psi^\pm) \geq -C_1\omega, \quad e^{\rho_\varepsilon(\psi^+-\psi^-)} \leq e^{-\rho_\varepsilon(\psi^-)+C_1},$$

where C_1 depends on C and the Lelong numbers of the currents $C\omega + dd^c\psi^\pm$. Now, for each $\varepsilon > 0$, it follows from [Yau78] that there exists a unique $\varphi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\sup_X \varphi_\varepsilon = 0$ and

$$(\omega + dd^c\varphi_\varepsilon)^n = c_\varepsilon e^{\rho_\varepsilon(\psi^+)-\rho_\varepsilon(\psi^-)}\omega^n = f_\varepsilon\omega^n,$$

where $c_\varepsilon > 0$ is a normalization constant. Since $e^{\rho_\varepsilon(\log f)}$ converges point-wise to f on X and since $e^{\rho_\varepsilon(\log f)} \leq \rho_\varepsilon(e^{\log f})$, by the *General Lebesgue Dominated Convergence Theorem* we see that $e^{\rho_\varepsilon(\log f)}$ converges to f in $L^1(X)$ as $\varepsilon \to 0$. This implies that c_ε converges to 1 as $\varepsilon \to 0$. Then we can assume that $c_\varepsilon \leq 2$. Thus we get the following uniform control

$$f_\varepsilon \leq e^{-\rho_\varepsilon(\psi^-)+C_2}.$$

By Lemma 4.2.5 below we know that φ_ε converges to φ in $L^1(X)$. Thus the set

$$\mathcal{U} := \{\varphi_\varepsilon \;\big|\; \varepsilon > 0\} \cup \{\varphi\}$$

is compact in $L^1(X)$. Then it follows from the uniform Skoda integrability theorem (Lemma 4.2.6 below) that for any $A > 0$ we have

$$\sup_{\varepsilon>0} \int_X e^{-A\varphi_\varepsilon} \omega^n < +\infty.$$

Thus, we can apply Theorem 4.2.3 to find $C_3 > 0$ under control such that

$$\Delta_\omega \varphi_\varepsilon \leq C_3 e^{-2\psi^-}.$$

Fix a compact $K \Subset X \setminus D$, $k \geq 2$ and $\beta \in (0,1)$. Now since $0 < f \in \mathcal{C}^\infty(X \setminus D)$ we have uniform controls on the derivatives of all orders of $\log f_\varepsilon$ on K. Using the standard Evans-Krylov method and Schauder estimates we then obtain

$$\|\varphi_\varepsilon\|_{\mathcal{C}^{k,\beta}(K)} \leq C_{K,k,\beta}.$$

This explains the smoothness of φ on $X \setminus D$.

Lemma 4.2.5. *Let (X,ω) be a compact Kähler manifold of dimension n. Let (f_j) be a sequence of non-negative functions on X such that $\int_X f_j \omega^n = \int_X \omega^n$. Assume that f_j converges in $L^1(X)$ and point-wise to f. For each j, let $\varphi_j \in \mathcal{E}(X,\omega)$ be the unique normalized solution to $\mathrm{MA}\,(\varphi_j) = f_j \omega^n$. Then φ_j converges in $L^1(X)$ to $\varphi \in \mathcal{E}(X,\omega)$ the unique normalized solution to $\mathrm{MA}\,(\varphi) = f\omega^n$.*

Proof. We can assume that φ_j converges in $L^1(X)$ to $\psi \in \mathrm{PSH}(X,\omega)$. It follows from the Hartogs lemma that $\sup_X \psi = 0$. For each $j \in \mathbb{N}$ set

$$\psi_j := \left(\sup_{k\geq j} \varphi_j\right)^* \text{ and } u_j := \max(\psi_j, \varphi - 1).$$

Then we see that $\psi_j \downarrow \psi$ and $u_j \downarrow u := \max(\psi, \varphi - 1) \in \mathcal{E}(X,\omega)$. We also have that $\sup_X u = 0$. It follows from the comparison principle that

$$\mathrm{MA}\,(u_j) \geq \min\left(f, \inf_{k\geq j} f_k\right) \omega^n = g_j \omega^n.$$

By the continuity of the Monge-Ampère operator along decreasing sequences in $\mathcal{E}(X,\omega)$ we get

$$\mathrm{MA}\,(u) = \lim_{j\to+\infty} \mathrm{MA}\,(u_j) \geq \lim_{j\to+\infty} g_j\omega^n = f\omega^n.$$

Then the equality holds since they have the same total mass. Finally, by the uniqueness result in the class $\mathcal{E}(X,\omega)$ (see [Din09]) we deduce that $u=\varphi$, which implies that $\psi=\varphi$. The proof is thus complete. □

By [GZ07], functions in $\mathcal{E}(X,\omega)$ have zero Lelong number at every point on X. Thus the following lemma is a direct consequence of the uniform Skoda integrability theorem due to Zeriahi [Zer01]:

Lemma 4.2.6. *Let $\mathcal{U}$ be a compact family of functions in $\mathcal{E}(X,\omega)$. Then for each $C_1>0$ there exists C_2 depending on C_1 and $\mathcal{U}$ such that*

$$\int_X e^{-C_1\phi}\omega^n \leq C_2,\ \forall\phi\in\mathcal{U}.$$

4.3 Asymptotic behavior near the divisor

In Theorem 4.2.1 we have given a very general $\mathcal{C}^0$ estimate. We only assumed that the density f is bounded by $e^{-\phi}$ for some quasi plurisubharmonic function ϕ, and there is no regularity assumption on D. It is therefore natural to investigate the asymptotic behavior of the solution near D when we have more information about D and about the behavior of f near D.

Let X be a compact Kähler manifold of dimension n and let ω be a Kähler form on X. Let $D=\sum_{j=1}^N D_j$ be a simple normal crossing divisor on X. Here "simple normal crossing" means that around each intersection point of k components $D_{j_1},...,D_{j_k}$ ($k\leq N$), we can find complex coordinates $z_1,...,z_n$ such that for each $l=1,...,k$ the hypersurface D_{j_l} is locally given by $z_l=0$. For each j, let L_j be the holomorphic line bundle defined by D_j. Let s_j be a holomorphic section of L_j defining D_j, i.e $D_j=\{s_j=0\}$. We fix a hermitian metric h_j on L_j such that $|s_j|:=|s_j|_{h_j}\leq 1/e$. We say that f satisfies Condition $\mathcal{S}(B,\alpha)$ for some $B>0,\alpha>0$ if

$$f \leq \frac{B}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^{1+\alpha}}. \tag{4.3.1}$$

4.3.1 Volume-capacity domination

Lemma 4.3.1. *Assume that f satisfies (4.3.1) for some $B>0,\alpha>0$. Then for each $0<\gamma<\alpha$ we can find $A>0$ which only depends on B,α,γ,ω such that*

$$\mathrm{vol}_f(E) := \int_E f\omega^n \leq A\,\mathrm{Cap}_\omega(E)^\gamma,\ \forall E\subset X,$$

where Cap_ω is the Monge-Ampère capacity introduced in [Kol03], [GZ05].

Before giving the proof of the lemma, let us recall the definition and basic facts about Cegrell's classes. We refer the reader to [Ceg98, Ceg04] for more details.

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. The class $\mathcal{E}_0(\Omega)$ consists of bounded psh functions which vanish on the boundary and have finite total mass.

We say that $u \in \mathcal{E}^p(\Omega), p > 0$ if there exists a sequence $(u_j) \subset \mathcal{E}_0(\Omega)$ decreasing to u such that

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty.$$

A function u belongs to $\mathcal{F}(\Omega)$ if there exists a sequence $(u_j) \subset \mathcal{E}_0(\Omega)$ decreasing to u such that

$$\sup_j \int_\Omega (dd^c u_j)^n < +\infty.$$

We recall the local Monge-Ampère capacity introduced in [BT82]: for any Borel subset $E \subset \Omega$, we define

$$\mathrm{Cap}_{\mathrm{BT}}(E,\Omega) := \sup \left\{ \int_E (dd^c u)^n \;\; \Big| \;\; u \in \mathrm{PSH}(\Omega) \; , \; -1 \leq u \leq 0 \right\}.$$

The relative extremal function of E with respect to Ω is

$$u_{E,\Omega} := \sup \left\{ u \in \mathrm{PSH}(\Omega) \;\; \Big| \;\; u \leq 0 \text{ on } \Omega \; , \; u \leq -1 \text{ on } E \right\}.$$

Proof of Lemma 4.3.1. It follows from [Koł03] that Cap_ω is comparable to the local capacity $\mathrm{Cap}_{\mathrm{BT}}(\cdot,\Omega)$, where Ω is an open subset contained in a local chart. By considering E a small subset contained in a local chart we reduce the problem to showing that

$$\mathrm{vol}_g(E) \leq A_1 \mathrm{Cap}_{\mathrm{BT}}(E,\mathbb{D}^n)^\alpha, \forall E \Subset \mathbb{D}^n_\delta \Subset \mathbb{D}^n, \tag{4.3.2}$$

where $\mathbb{D}^n$ is the unit polydisk in $\mathbb{C}^n$, $\delta > 0$ small enough and fixed, and

$$g(z) = g(z_1, ..., z_n) := \frac{1}{\prod_{j=1}^k |z_j|^2 (1 - \log |z_j|)^{1+\alpha}}, k \leq n.$$

We prove (4.3.2) by induction using the ideas in [ACK+09]. We start with the case $n = 1$. Set $E_r := E \cap \partial \mathbb{D}_r$, for any $r \in [0,t]$. Define now $\tilde{E} := \{r \in [0,t] \,|\, E_r \neq \emptyset\}$ and denote by $l(\tilde{E})$ the length of $\tilde{E}$. Since the function $r \mapsto \frac{1}{r(1-\log r)^{1+\alpha}}$ is non-increasing when r is small, we obtain

$$\begin{aligned}\int_E g(z)dV(z) &= \int_0^{2\pi}\int_{\tilde{E}} \frac{drd\theta}{r(1-\log r)^{1+\alpha}} \\ &\leq 2\pi \int_0^{\ell(\tilde{E})} \frac{dr}{r(1-\log r)^{1+\alpha}} \\ &\leq \frac{C_1}{(-\log l(\tilde{E}))^{\alpha}} \\ &\leq C_2 \left[\mathrm{Cap}_{\mathrm{BT}}(E,\mathbb{D})\right]^{\alpha},\end{aligned}$$

where the last inequality follows from [Koł94, p.1336]. Assume that the result holds for $n-1$. Let us prove it for n. Without loss of generality we can assume that E is compact in $\mathbb{D}^n$. We can also assume that $k = n$ (if $k < n$ the situation is much easier). Set $h = h^*_{E,\mathbb{D}^n}$ the relative extremal function of E. Consider

$$g_n(w) := \frac{1}{|w|^2(1-\log|w|)^{1+\alpha}}\ ,\ g_{n-1}(z) := \frac{1}{\prod_{j=1}^{n-1}|z_j|^2(1-\log|z_j|)^{1+\alpha}}.$$

For each $w \in \mathbb{D}$ set

$$E_w = \{z \in \mathbb{D}^{n-1} \mid h(z,w) \leq -1\} \text{ and } h_w = h(\cdot, w).$$

By induction hypothesis we get

$$\begin{aligned}\mathrm{vol}_g(E) &= \int_{\mathbb{D}} \mathrm{vol}_{g_{n-1}}(E_w) g_n(w) dV_2(w) \\ &\leq A_1 \int_{\mathbb{D}} \left[\mathrm{Cap}_{\mathrm{BT}}(E_w, \mathbb{D}^{n-1})\right]^{\gamma} g_n(w) dV(w).\end{aligned}$$

Fix now $w \in \mathbb{D}$ and denote by $u = h^*_{E_w,\mathbb{D}}$ the relative extremal function of E_w. Since $h \in \mathcal{F}(\mathbb{D}^n)$ it follows from [ACK$^+$09, Theorem 3.1] that $h_w \in \mathcal{E}^1(\mathbb{D}^{n-1})$. We also have $h_w \leq u$ and $h_w = -1$ on E_w. Using integration by parts we get

$$\begin{aligned}\mathrm{Cap}_{\mathrm{BT}}(E_w, \mathbb{D}^{n-1}) &\leq \int_{\mathbb{D}^{n-1}} (-h_w)(dd^c u)^{n-1} \\ &\leq \int_{\mathbb{D}^{n-1}} (-h_w)(dd^c h_w)^{n-1} =: -\varphi(w).\end{aligned}$$

By [ACK$^+$09, Theorem 3.1] we know that $\varphi \in \mathcal{F}(\mathbb{D})$. Moreover, we also have $\varphi \geq -A_0$ for some universal constant A_0 (here A_0 depends on δ). Indeed, let v be the relative extremal function of $\mathbb{D}^n_\delta$ with respect to $\mathbb{D}^n$. Since $h \geq v$, it is easy to see that for each $w \in \mathbb{D}$, $h_w \geq v_w$. From this we get a uniform lower bound for φ. Since E is compact in $\mathbb{D}^n$ we also get

$$\mu = \int_{\mathbb{D}} dd^c\varphi = \int_{\mathbb{D}^n} (dd^c h)^n = \mathrm{Cap}_{\mathrm{BT}}(E, \mathbb{D}^n).$$

Thus, using the previous part (when $n = 1$) we obtain

$$\begin{aligned}
\mathrm{vol}_g(E) &\leq A_1 \int_{\mathbb{D}} (-\varphi(w))^{\gamma} g_n(w) dV_2(w) \\
&= A_2 \int_0^{A_0} t^{\gamma-1} \, \mathrm{vol}_{g_n}(\varphi < -t) dt \\
&\leq A_3 \int_0^{A_0} t^{\gamma-\beta_1-1} \mu^{\beta_1} dt \\
&= A_4 \left[\mathrm{Cap}_{\mathrm{BT}}(E, \mathbb{D}^n)\right]^{\beta_1}.
\end{aligned}$$

Here, we choose $\beta_1 < \gamma$ so that the integrals converge. In the above we have used the fact that

$$\mathrm{Cap}_{\mathrm{BT}}(v < -t) \leq \frac{1}{t} \int_{\mathbb{D}} dd^c v, \ \forall v \in \mathcal{F}(\mathbb{D}), \ \forall t > 0.$$

Since β_1 can be chosen arbitrarily near γ (and the constant A_4 will increase), the result follows. □

When $\alpha = 1$ we get the following estimate.

Lemma 4.3.2. *Let* $\mu = f\omega^n$, $f = \frac{1}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^2}$. *Then there exists* $A > 0$ *such that for every Borel subset* $E \subset X$ *we have*

$$\mu(E) \leq A \cdot \left[\eta + (-\log \eta)^n \mathrm{Cap}_{\omega}(E)\right], \ \forall \eta \in (0, 1/e). \tag{4.3.3}$$

Proof. We only give a sketch of the proof since it is essentially a copy of the proof of Lemma 4.3.1 with a small change. We also use the same notation as there. Without loss of generality we can assume that $E \Subset \mathbb{D}_\delta^n \Subset \mathbb{D}^n$ for some small fixed δ. The function φ belongs to $\mathcal{F}(\mathbb{D})$. The same arguments as in Lemma 4.3.1 show that φ is also bounded from below by $-A_1$ for some universal constant $A_1 > 0$. In the final step we get

$$\begin{aligned}
\mathrm{vol}_g(E) &\leq A_2 \int_{\mathbb{D}} \left(\eta + (-\log\eta)^{n-1}(-\varphi(w))\right) g_n(w) dV_2(w) \\
&= A_3\eta + A_2(-\log\eta)^{n-1} \int_0^{A_1} \mathrm{vol}_{g_n}(\varphi < -t) dt \\
&\leq A_3\eta + A_4\eta^2(-\log\eta)^{n-1} + A_5(-\log\eta)^{n-1} \int_{\eta^2}^{A_1} \mathrm{Cap}_{\omega}(\varphi < -t) dt \\
&\leq A_6\eta + A_5(-\log\eta)^{n-1} \int_{\eta^2}^{A_1} \frac{1}{t} \left[\int_{\mathbb{D}} dd^c\varphi\right] dt \\
&\leq A_6\eta + A_7(-\log\eta)^n \int_{\mathbb{D}} dd^c\varphi.
\end{aligned}$$

□

Lemma 4.3.3. *Let* $\varphi \in \mathcal{E}(X,\omega)$ *be such that* $\sup_X \varphi = 0$ *and assume* $\mu = \mathrm{MA}\,(\varphi) \leq A\mathrm{Cap}_\omega$ *for some* $A > 0$. *Then there exists* $C, c > 0$ *depending on* A *such that*

$$\mathrm{Cap}_\omega(\varphi < -t) \leq Ce^{-ct}, \ \forall t > 0,$$

In particular, if $\beta < c$ *then* $\int_X e^{-\beta\varphi} d\mu \leq C'$, *with* $C' = C(\beta, A) > 0$.

Proof. Fix $s, t > 1$. By standard application of the comparison principle we get

$$\begin{aligned}
\mathrm{Cap}_\omega(\varphi < -t - s) &\leq \int_{\{\varphi<-t\}} \left(\omega + \frac{1}{s} dd^c\varphi\right)^n \qquad (4.3.4)\\
&\leq \frac{1}{s^n}\int_{\{\varphi<-t\}} \sum_{k=0}^{n} C_n^k (s-1)^k \omega^k \wedge \omega_\varphi^{n-k}\\
&\leq \int_{\{\varphi<-t\}} \omega^n + \frac{2^n}{s}\int_{\{\varphi<-t\}} \mathrm{MA}\,(\varphi),
\end{aligned}$$

where the last inequality follows from the partial comparison principle (see [Din09, Theorem 2.3]). It follows from [GZ05] that

$$\int_{\{\varphi<-t\}} \omega^n \leq C_1 e^{-at}, a > 0.$$

Choose $s := 2^n Ae$ and fix $\varepsilon < \min(1, a, 1/s)$. Set

$$F(t) := e^{\varepsilon t}\mathrm{Cap}_\omega(\varphi < -t), \ t \geq 1.$$

Now, since $\mu \leq A\mathrm{Cap}_\omega$, from (4.3.4) we get

$$F(t+s) \leq C_2 + bF(t),$$

where $b = 2^n Ae^{\varepsilon s}/s < 1$. This yields $\sup_{t\geq 1} F(t) \leq C_3$, for some $C_3 > 0$ depending on A. We finally get

$$\mathrm{Cap}_\omega(\varphi < -t) \leq Ce^{-ct}, c < \varepsilon.$$

The last statement easily follows since it follows from [BGZ08, Lemma 2.3] that

$$\int_{\{\varphi<-t\}} \mathrm{MA}\,(\varphi) \leq t^n \mathrm{Cap}_\omega(\varphi < -t), \quad \forall t \geq 1.$$

□

4.3.2 Proof of Theorem 3

Assume in this section that f satisfies Condition $\mathcal{S}(B, \alpha)$ for some $B > 0, \alpha > 0$. We consider three cases depending on the value of α.

The case when $\alpha > 1$

The continuity of φ and the $\mathcal{C}^0$ estimate follow directly from Lemma 4.3.1 and Kołodziej's classical result (see [Koł98]).

The case when $0 < \alpha < 1$

Fix $\beta \in (1-\alpha, 1)$ and set $\delta = \alpha + \beta - 1$, and

$$u_\beta := \sum_{j=1}^{N} -a(-\log|s_j|)^\beta,$$

where $a > 0$ is small enough so that $u_\beta \in \mathrm{PSH}(X, \omega)$. By Theorem 4.2.1 we have

$$\varphi \geq \sum_{j=1}^{N} \log|s_j| - C_0,$$

for some positive constant C_0 depending on B. By simple computations we obtain

$$\mathrm{MA}\,(\varphi) \leq \frac{C_1 f_{1-\beta}\omega^n}{(-\varphi)^\delta},$$

for some positive constant C_1 depending on C_0. Here for each $r > 0$, we set

$$f_r := \frac{1}{\prod_{j=1}^{N} |s_j|^2(-\log|s_j|)^{1+r}}.$$

We also get

$$\mathrm{MA}\,(u_\beta - C_2) \geq \frac{C_1 f_{1-\beta}\omega^n}{(-u_\beta + C_2)^\delta},$$

where $C_2 > 0$ depends on C_1, δ. The comparison principle yields that $\varphi \geq u_\beta - C_2$.

The case when $\alpha = 1$

Assume D is a smooth divisor. Consider the model function

$$\psi := -A_1 \sum_{j=1}^{N} \log(-\log|s_j| + A_2),$$

where $A_1 > 0$ is big and A_2 is chosen so that ψ is $\omega/2$-psh on X. From the first step in Lemma 4.3.1 we get that there exists a constant $A > 0$ such that $\mathrm{vol}_f(E) \leq A\mathrm{Cap}_\omega(E)$, for any $E \subset X$. Then it follows from Lemma 4.3.3

that $\int_X e^{-c\varphi} f\omega^n < C_1$ for some small constant $c > 0$ depending on B. Here C_1 depends on c and B. Thus, for $t > 0$, $p > 1$, by Hölder inequality we get

$$\begin{aligned}
\int_{\{\varphi<\psi-t\}} \mathrm{MA}\,(\varphi) &\leq \int_{\{\varphi<\psi-t\}} e^{-c\varphi/p} e^{c\psi/p} f\omega^n \\
&\leq \left(\int_X e^{-c\varphi} f\omega^n\right)^{1/p} \left(\int_{\{\varphi<\psi-t\}} e^{c\psi/(p-1)} f\omega^n\right)^{1-1/p} \\
&\leq C_2 \left(\mathrm{Cap}_\psi(\varphi < \psi - t)\right)^\gamma,
\end{aligned}$$

where $\gamma < A_1 c/p + (p-1)/p$ and $C_2 > 0$ is a universal constant. The last inequality follows from the volume-capacity domination (Lemma 4.3.1) and from Lemma 4.1.7. Now if $A_1 c > 1$ we can choose $\gamma > 1$ and the result follows as in Theorem 4.2.1.

4.3.3 Regularity near the divisor D

In this subsection we will discuss about the behavior of the solution to equation (4.0.1) near the divisor D. We prove the following result when $\alpha < 1$.

Proposition 4.3.4. *Consider $f = \frac{h}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^{1+\alpha}}$, where $1/B \leq h \leq B$ on X and $\alpha \in (0,1)$. Assume that f is normalized so that $\int_X f\omega^n = \int_X \omega^n$. Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique normalized solution of (4.0.1). Then for each $0 < p < 1-\alpha$ and each $1-\alpha < q < 1$, we have*

$$-a_1 \sum_{j=1}^N (-\log|s_j|)^q - A_1 \leq \varphi \leq -a_2 \sum_{j=1}^N (-\log|s_j|)^p + A_2,$$

where $a_1, A_1 > 0$ depend on B, α, q while $a_2, A_2 > 0$ depend on B, α, p. In particular, the solution φ goes to $-\infty$ on D.

Proof. One inequality has been proved in Section 4.3.2. Let us prove the upper bound. We normalize φ such that $\sup_X \varphi = -1$. Fix $p \in (0, 1-\alpha)$ set $\delta := (1-\alpha-p)/p > 0$.
Consider $u_p := -\sum_{j=1}^N a_2(-\log|s_j|)^p$, where $a_2 > 0$ is small so that u_p is ω-psh on X. Then we can find $C_3 > 0$ such that

$$\mathrm{MA}\,(u_p) \leq \frac{C_3 f\omega^n}{(-u_p)^\delta},$$

while since $\varphi \leq 0$, for some $A_2 > 0$ big enough (for instance $A_2^\delta = C_3$) we have

$$\mathrm{MA}\,(\varphi - A_2) \geq \frac{C_3 f\omega^n}{(-\varphi + A_2)^\delta}.$$

The comparison principle then yields the desired upper bound. □

In the same way we obtain a similar upper bound when $\alpha = 1$.

Proposition 4.3.5. *Assume that f is normalized so that $\int_X f\omega^n = \int_X \omega^n$ and*

$$f \geq \frac{1}{B\prod_{j=1}^N |s_j|^2(-\log|s_j|)^2}.$$

Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique normalized solution of (4.0.1). Then for any $p \in (0,1)$ there exist $a, A > 0$ depending on B, p such that

$$\varphi \leq -a\sum_j [\log(-\log|s_j|)]^p + A.$$

In particular, φ is not bounded and goes to $-\infty$ on D.

Proof. The proof uses the same arguments as in Proposition 4.3.4. □

4.4 The case of semipositive and big classes

In this section we prove Theorem 4. For convenience let us recall the setting. We assume that (X,ω) is a compact Kähler manifold of dimension n and D is an arbitrary divisor on X. Let $E = \sum_{j=1}^M a_j E_j$ be an effective snc divisor on X. Let θ be a smooth semipositive form on X such that $\int_X \theta^n > 0$ and $\{\theta\} - c_1(E)$ is ample. Consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = f\omega^n, \tag{4.4.1}$$

where $0 \leq f \in L^1(X,\omega^n)$ satisfies the compatibility condition $\int_X f\omega^n = \int_X \theta^n$.

For each $j = 1, ..., M$ let K_j be the holomorphic line bundle defined by E_j. Let σ_j be a holomorphic section of K_j that vanish on E_j. We fix hermitian metric h_j on K_j such that $|\sigma_j| \leq 1/e$. Since $\{\theta\} - c_1(E)$ is ample, we can assume that

$$\theta + dd^c\phi = \omega_0 + [E],$$

where ω_0 is a Kähler form on X and

$$\phi := \sum_{j=1}^M a_j \log|\sigma_j|.$$

By rescaling ω we can also assume that $\omega_0 \geq \omega$. Recall that f satisfies Condition $\mathcal{H}_f$ on X, i.e. there is a constant $C > 0$ such that

$$f = e^{\psi^+ - \psi^-},\ \ dd^c\psi^{\pm} \geq -C\omega,\ \sup_X \psi^+ \leq C,\ \ \psi^- \in L^\infty_{\rm loc}(X \setminus D). \tag{4.4.2}$$

4.4.1 Uniform estimate

The following $\mathcal{C}^0$-lower bound can be proved in the same ways as we have done in Theorem 4.2.1:

Theorem 4.4.1. *Assume that D, E and θ are as above and f satisfies (4.4.2). Let φ be the unique normalized solution to equation (4.4.1). Then φ is uniformly bounded away from $D \cup E$. More precisely, for any $a > 0$ there exists $A > 0$ depending on C and $\int_X e^{-2\varphi/a}\omega^n$ such that*

$$\varphi \geq a\psi^- + \phi - A.$$

Proof. Fix $a > 0$ very small so that

$$\psi := a\psi^- + \frac{1}{2}\phi \in \mathrm{PSH}(X, \theta/2).$$

It follows from Proposition 3.1 in [EGZ09] that

$$\mathrm{vol}_\omega \leq C_1 \exp\left(\frac{-C_2}{\left[\mathrm{Cap}_{\theta/2}\right]^{1/n}}\right),$$

for some universal constants $C_1, C_2 > 0$. Now, the same proof of Lemma 4.1.7 yields

$$\mathrm{Cap}_{\theta/2} \leq \mathrm{Cap}_\psi,$$

where Cap_ψ is the generalized capacity defined by the form θ and ψ:

$$\mathrm{Cap}_\psi(E) := \sup\left\{\int_E (\theta + dd^c u)^n \;\middle|\; u \in \mathrm{PSH}(X, \theta),\ \psi - 1 \leq u \leq \psi\right\}.$$

Then we can repeat the arguments in the proof of Theorem 4.2.1 to get the result. □

4.4.2 Laplacian estimate

We now prove a C^2 a priori estimate in the semipositive and big case. Even when f is smooth on X, φ is only smooth in the ample locus of θ. To get rid of this, we replace θ by $\theta + t\omega$, $t > 0$. In principle, the $\mathcal{C}^2$ estimate will depends heavily on $t > 0$ and we will have serious problem when $t \downarrow 0$. But, fortunately, the so-called Tsuji's trick (see [Tsu88]) allows us to get around this difficulty. In the sequel, we follow essentially the ideas in [BEGZ10].

Theorem 4.4.2. *Let $f = e^{\psi^+ - \psi^-}$ where ψ^+, ψ^- are smooth on X. Fix $t \in (0, 1)$. Let $\varphi \in \mathcal{C}^\infty(X)$ be the unique normalized solution to*

$$(\theta + t\omega + dd^c\varphi)^n = e^{\psi^+ - \psi^-}\omega^n.$$

Assume given a constant $C > 0$ such that

$$dd^c\psi^{\pm} \geq -C\omega, \quad \sup_X \psi^+ \leq C.$$

Assume also that the holomorphic bisectional curvature of ω is bounded from below by $-C$. Then there exists $A > 0$ depending on C and $\int_X e^{-2(4C+1)\varphi}\omega^n$ such that

$$\Delta_\omega \varphi \leq Ae^{-2\psi^- -(4C+1)\phi}.$$

Proof. Ignoring the dependence on t, we denote $\omega_\varphi := \theta + t\omega + dd^c\varphi$. Consider the following function

$$H := \log \mathrm{tr}_\omega(\omega_\varphi) + 2\psi^- - (4C+1)(\varphi - \phi),$$

Since ϕ goes to $-\infty$ on E, we see that H attains its maximum on $X \setminus E$ at some point $x_0 \in X \setminus E$. From now on we carry all computations on $X \setminus E$. We can argue as in Theorem 4.2.3 to obtain

$$\Delta_{\omega_\varphi} \log \mathrm{tr}_\omega(\omega_\varphi) \geq -3C\mathrm{tr}_{\omega_\varphi}(\omega) - \Delta_{\omega_\varphi}\psi^-. \tag{4.4.3}$$

Since $\omega_0 + t\omega \geq \omega$ we get

$$\Delta_{\omega_\varphi}(\varphi - \phi) \leq \mathrm{tr}_{\omega_\varphi}(\omega_\varphi - \omega_0 - t\omega) \leq n - \mathrm{tr}_{\omega_\varphi}(\omega). \tag{4.4.4}$$

Therefore, from (4.4.3) and (4.4.4) we deduce that on $X \setminus E$

$$\Delta_{\omega_\varphi} H \quad \geq \quad \mathrm{tr}_{\omega_\varphi}(\omega) - n(4C+1).$$

We now apply the maximum principle to the function H at x_0:

$$0 \geq \Delta_{\omega_\varphi} H(x_0) \geq \mathrm{tr}_{\omega_\varphi}(\omega)(x_0) - n(4C+1).$$

Furthermore, by Lemma 4.2.4 we get

$$\mathrm{tr}_\omega(\omega_\varphi)(x_0) \leq ne^{\psi^+ - \psi^-}(x_0)\left(\mathrm{tr}_{\omega_\varphi}(\omega)\right)^{n-1}(x_0) \leq A_1 e^{\psi^+ - \psi^-}(x_0),$$

and hence, since $\sup_X \psi^+ \leq C$,

$$\log \mathrm{tr}_\omega(\omega_\varphi)(x_0) \leq \log A_1 + \psi^+(x_0) - \psi^-(x_0) \leq A_2 - \psi^-(x_0)\,.$$

It follows that

$$H(x) \leq H(x_0) \leq A_2 + \psi^-(x_0) - (4C+1)(\varphi - \phi)(x_0).$$

By assumption and the $\mathcal{C}^0$ estimate in Theorem 4.4.1 we have

$$\varphi \geq \frac{1}{4C+1}\psi^- + \phi - A_3,$$

where A_3 depends on C and $\int_X e^{-2(4C+1)\varphi}\omega^n$. Thus

$$\log \mathrm{tr}_\omega(\omega_\varphi) \leq A_4 - 2\psi^- + (4C+1)(\varphi - \phi).$$

We finally get

$$\mathrm{tr}_\omega(\omega_\varphi) \leq A_5 e^{-2\psi^- -(4C+1)\phi}.$$

□

Proof of Theorem 4. We proceed as in Section 4.2.3. We also borrow the notations there. Let $\rho_\varepsilon(\psi^\pm)$ be the Demailly's smoothing regularization of $\psi^\pm$. For each $\varepsilon > 0$ it follows from [Yau78] that there exists a unique $\varphi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\sup_X \varphi_\varepsilon = 0$ and

$$(\theta + \varepsilon\omega + dd^c\varphi_\varepsilon)^n = c_\varepsilon e^{\rho_\varepsilon(\psi^+)-\rho_\varepsilon(\psi^-)}\omega^n,$$

where c_ε is a normalization constant. As in Section 4.2.3 we have a uniform control on the right-hand side:

$$c_\varepsilon e^{\rho_\varepsilon(\psi^+)-\rho_\varepsilon(\psi^-)} \leq e^{C-\psi_\varepsilon^-}.$$

Now, we can copy the arguments in Section 4.2.3 since our uniform estimate and laplacian estimate do not depend on ε. The proof is thus complete. □

Chapter 5

Generalized Monge-Ampère Capacities

Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n and let D be an arbitrary divisor on X. Consider the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\omega^n, \tag{5.0.1}$$

where $0 \leq f \in L^1(X)$ is such that $\int_X f\omega^n = \int_X \omega^n$. It follows from [GZ07] and [Din09] that equation (5.0.1) has a unique normalized solution in the finite energy class $\mathcal{E}(X, \omega)$. We say that the solution φ is normalized if $\sup_X \varphi = 0$.

If f is strictly positive and smooth on X, we know from the seminal paper of Yau [Yau78] that the solution is also smooth on X. Recall that this solves in particular the Calabi conjecture and allows to construct Ricci flat metrics on X whenever $c_1(X) = 0$.

Given f positive and smooth on $X \setminus D$, it is natural to investigate the regularity of the solution. In [DNL14a] we have proved in many cases that the solution φ is smooth in $X \setminus D$.

As in the classical case of Yau [Yau78], the most difficult step is to establish an a priori $\mathcal{C}^0$-estimate. This estimate is much more difficult in our situation since in general the solution is not globally bounded. A natural idea is to bound the normalized solution from below by a singular quasi plurisubharmonic function (qpsh for short). This is where generalized Monge-Ampère capacities play a crucial role.

We recall the notion of the classical capacity Cap_ω introduced and studied in [KołO3] and [GZ05]:

$$\mathrm{Cap}_\omega(E) = \sup\left\{\int_E (\omega + dd^c u)^n \ \middle|\ u \in \mathrm{PSH}(X, \omega),\ -1 \leq u \leq 0\right\},\ E \subset X.$$

A strong comparison between the Lebesgue measure and Cap_ω, as is needed in a celebrated method due to Kołodziej [Koł98], does not hold in our setting. We therefore study other capacities to provide an a priori $\mathcal{C}^0$-estimate. In dealing with complex Monge-Ampère equations in quasiprojective varieties we were naturally lead to work with generalized capacities of type $\mathrm{Cap}_{\psi-1,\psi}$ in [DNL14a] (see below for their definition).

In this paper, we make a systematic study of these capacities as well as the more general $\mathrm{Cap}_{\varphi,\psi}$ capacities: let φ, ψ be two ω-plurisubharmonic functions on X such that $\varphi < \psi$ on X modulo possibly a pluripolar set. The (φ,ψ)-Capacity of a Borel subset $E \subset X$ is defined by

$$\mathrm{Cap}_{\varphi,\psi}(E) := \sup\left\{ \int_E (\omega + dd^c u)^n \ \middle|\ u \in \mathrm{PSH}(X,\omega),\ \varphi \le u \le \psi \right\}.$$

Here, for a ω-psh function u, $(\omega+dd^c u)^n$ is the non-pluripolar Monge-Ampère measure of u. See Section 2 for the definition. When $\varphi \equiv \psi - 1$, we drop the index φ and denote the $(\psi-1,\psi)$-Capacity by Cap_ψ,

$$\mathrm{Cap}_\psi := \mathrm{Cap}_{\psi-1,\psi}.$$

This is exactly the generalized capacity used in our previous paper [DNL14a]. If moreover ψ is constant, $\psi \equiv C$, we recover the Monge-Ampère capacity defined above

$$\mathrm{Cap}_C = \mathrm{Cap}_\omega.$$

Given any subset $E \subset X$, we define the outer (φ,ψ)-capacity of E by

$$\mathrm{Cap}^*_{\varphi,\psi}(E) := \inf\left\{ \mathrm{Cap}_{\varphi,\psi}(U) \ \middle|\ U \text{ is an open subset of } X,\ E \subset U \right\}.$$

We say that the (φ,ψ)-capacity *characterizes pluripolar sets* on X if for any subset $E \subset X$, the following holds

$$\mathrm{Cap}^*_{\varphi,\psi}(E) = 0 \Longleftrightarrow \text{ E is a pluripolar subset of } X.$$

If $E \subset X$ is a Borel subset we set

$$h_{\varphi,\psi,E}(x) := \sup\left\{ u(x) \ \middle|\ u \in \mathrm{PSH}(X,\omega), u \le \psi \text{ on } X,\ u \le \varphi \text{ q.e. } E \right\}.$$

Here, quasi everywhere (q.e. for short) means outside a pluripolar set. Let $h^*_{\varphi,\psi,E}$ be its upper semicontinuous regularization which we call the (φ,ψ)-extremal function of E. We establish a useful characterization of the (φ,ψ)-capacity in terms of the relative extremal function for any subset.

When φ belong to the finite energy class $\mathcal{E}(X,\omega)$ we can bound $\mathrm{Cap}_{\varphi,\psi}$ by $F(\mathrm{Cap}_\omega)$ for some positive function F which vanishes at 0. This uniform control turns out to be very useful in studying convergence of the complex Monge-Ampère operator since it allows us to replace quasi-continuous functions by continuous ones without affecting the final result. We also prove that the generalized Monge-Ampère capacity $\mathrm{Cap}_{\varphi,\psi}$ characterizes pluripolar sets when the lower weight is in $\mathcal{E}(X,\omega)$:

Theorem A. *Assume that* $\varphi \in \mathcal{E}(X,\omega)$ *and* $\psi \in \mathrm{PSH}(X,\omega)$ *such that* $\varphi < \psi$ *modulo a pluripolar subset.*

(i) *Let* $E \subset X$ *be a Borel subset of* X*, and denote by* h_E *the* (φ,ψ)*-extremal function of* E*. The outer* (φ,ψ)*-capacity of* E *is given by*

$$\mathrm{Cap}^*_{\varphi,\psi}(E) = \int_{\{h_E<\varphi\}} \mathrm{MA}\,(h_E) = \int_X \left(\frac{-h_E+\psi}{-\varphi+\psi}\right) \mathrm{MA}\,(h_E),$$

where $h_E := h^*_{\varphi,\psi,E}$ is the (φ,ψ)-extremal function of E.

(ii) *There exists a function* $F : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_{t\to 0^+} F(t) = 0$ *and such that for all Borel subset* E*,*

$$\mathrm{Cap}_{\varphi,\psi}(E) \leq F(\mathrm{Cap}_\omega(E)).$$

(iii) $\mathrm{Cap}_{\varphi,\psi}$ *characterizes pluripolar sets.*

We stress that the function F in *(ii)* is quite explicit (see Theorem 5.1.9).

As we have underlined, these generalized capacities play an important role in studying complex Monge-Ampère equations on quasi-projective varieties (see [DNL14a]). We give in the second part of this paper several other applications.

We consider the following complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\omega^n,\ \lambda \in \mathbb{R}. \tag{5.0.2}$$

Assume that $0 < f \in \mathcal{C}^\infty(X \setminus D)$ satisfies Condition $\mathcal{H}_f$, i.e. f can be written as

$$f = e^{\psi^+ - \psi^-},\ \ \psi^\pm \text{ are quasi psh functions on } X\ ,\ \psi^- \in L^\infty_{\mathrm{loc}}(X \setminus D).$$

When $\lambda = 0$ and f satisfies $\int_X f\omega^n = \int_X \omega^n$, we proved in [DNL14a] that there is a unique normalized solution in $\mathcal{E}(X,\omega)$ which is smooth on $X \setminus D$. When $\lambda > 0$ and $\int_X f\omega^n < +\infty$ the same result holds since the $\mathcal{C}^0$ estimate follows easily from the comparison principle.

Consider now the case when $\lambda < 0$. In this case solutions do not always exist and when they do, there may be many of them. Our result here says that any solution in $\mathcal{E}(X,\omega)$ (*if exists*) is smooth on $X \setminus D$.

Theorem B. *Let* $0 < f \in \mathcal{C}^\infty(X \setminus D) \cap L^1(X)$*. Assume that* f *satisfies Condition* $\mathcal{H}_f$ *and* $\varphi \in \mathcal{E}(X,\omega)$ *is a solution of*

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\omega^n,\ \lambda < 0.$$

Then φ *is smooth on* $X \setminus D$*.*

Note that when $\lambda < 0$ and equation (5.0.2) has a solution in $\mathcal{E}(X,\omega)$, the measure $\mu = f\omega^n$ is dominated by MA (u) for some $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. In particular, $f \in L^1(X)$.

We next investigate the case when $\lambda > 0$ and f is not integrable on X. Of course solutions do not always exist. But observe that when φ is singular enough $e^\varphi f$ will be integrable on X and it is then reasonable to find a solution. For example, one can look at densities of the type

$$f \simeq \frac{1}{|s|^2},$$

which is not integrable. Here s is a holomorphic section of the line bundle associated to D. Such densities have been considered by Berman and Guenancia in their study of the compactification of the moduli space of canonically polarized manifolds [BG13]. They have shown that there exists a unique solution $\varphi \in \mathcal{E}(X,\omega)$ which is smooth in $X \setminus D$. As another application of the generalized Monge-Ampère capacities we show in the following result that in a general context whenever a solution in $\mathcal{E}(X,\omega)$ exists it is smooth outside D.

Theorem C. *Assume* $0 < f \in \mathcal{C}^\infty(X \setminus D)$ *satisfies Condition* $\mathcal{H}_f$. *If the equation*

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\omega^n, \ \lambda > 0$$

admits a solution $\varphi \in \mathcal{E}(X,\omega)$ *then* φ *is smooth on* $X \setminus D$.

Let us stress that in Theorem C we do not assume that $\int_X f\omega^n < +\infty$. It turns out that the existence of solutions in $\mathcal{E}(X,\omega)$ is equivalent to the existence of subsolutions in this class, these are easy to construct in concrete situations (see Example 5.3.7). We also obtain a similar result in the case of semipositive and big classes (see Theorem 5.3.8 and Example 5.3.9).

Finally we use generalized capacitites to study the critical integrability of a given $\phi \in \mathrm{PSH}(X,\omega)$.

Theorem D. *Let* $\phi \in \mathrm{PSH}(X,\omega)$ *and* $\alpha = \alpha(\phi) \in (0,+\infty)$ *be the canonical threshold of* ϕ, *i.e.*

$$\alpha = \alpha(\phi) := \sup\{t > 0 \ \mid \ e^{-t\phi} \in L^1(X)\}.$$

Then there exists $u \in \mathrm{PSH}(X,\omega)$ *with zero Lelong number at all points such that* $e^{u-\alpha\phi}$ *is integrable. Moreover, there exists a unique* $\varphi \in \mathcal{E}(X,\omega)$ *such that*

$$(\omega + dd^c\varphi)^n = e^{\varphi-\alpha\phi}\omega^n.$$

It turns out that one can even chose $u = \chi \circ \phi$ in $\mathcal{E}(X,\omega)$, as an explicit function of ϕ with attenuated singularities (see Theorem 5.3.10).

The paper is organized as follows. In section 2 we recall some known facts on energy classes, we introduce generalized capacities on compact

Kähler manifolds and prove Theorem A. As an application of the generalized capacities we give another proof of the domination principle in $\mathcal{E}(X,\omega)$ in Section 3. In Section 4 we use generalized capacities to study complex Monge-Ampère equations as (5.0.2). The proof of Theorem D will be given in Section 4 as well.

5.1 Generalized Monge-Ampère Capacities

Let (X,ω) be a compact Kähler manifold of complex dimension n. In this section we prove some basic properties of the (φ,ψ)-capacity and of the relative (φ,ψ)-extremal functions.

5.1.1 Energy classes

Definition 5.1.1. We let $\mathrm{PSH}(X,\omega)$ denote the class of ω-plurisubharmonic functions (ω-psh for short) on X, i.e. the class of functions φ such that locally $\varphi = \rho + u$, where ρ is a local potential of ω and u is a plurisubharmonic function.

Let φ be some unbounded ω-psh function on X and consider $\varphi_j := \max(\varphi,-j)$ the "canonical approximants". It has been shown in [GZ07] that

$$\mathbf{1}_{\{\varphi_j>-j\}}(\omega+dd^c\varphi_j)^n$$

is a non-decreasing sequence of Borel measures. We denote its limit by

$$\mathrm{MA}\,(\varphi) = (\omega+dd^c\varphi)^n := \lim_{j\to+\infty} \mathbf{1}_{\{\varphi_j>-j\}}(\omega+dd^c\varphi_j)^n.$$

Definition 5.1.2. We denote by $\mathcal{E}(X,\omega)$ the set of ω-psh functions having full Monge-Ampère mass:

$$\mathcal{E}(X,\omega) := \left\{\varphi\in \mathrm{PSH}(X,\omega) \;\mid\; \int_X \mathrm{MA}\,(\varphi) = \int_X \omega^n\right\}.$$

Let us stress that ω-psh functions with full Monge-Ampère mass have mild singularities. In particular, any $\varphi\in\mathcal{E}(X,\omega)$ has zero Lelong numbers $\nu(\varphi,\cdot)=0$ (see [GZ07, Corollary 1.8]). We also recall that, for every $\varphi\in\mathcal{E}(X,\omega)$ and any $\psi\in\mathrm{PSH}(X,\omega)$, the *generalized comparison principle* is valid, namely

$$\int_{\{\varphi<\psi\}}(\omega+dd^c\psi)^n \le \int_{\{\varphi<\psi\}}(\omega+dd^c\varphi)^n.$$

Definition 5.1.3. Let $\chi:\mathbb{R}^-\to\mathbb{R}^-$ be an increasing function such that $\chi(0)=0$ and $\chi(-\infty)=-\infty$. We denote by $\mathcal{E}_\chi(X,\omega)$ the class of ω-psh functions having finite χ-energy:

$$\mathcal{E}_\chi(X,\omega) := \left\{\varphi\in\mathcal{E}(X,\omega) \;\mid\; \chi(-|\varphi|)\in L^1(\mathrm{MA}\,(\varphi))\right\}.$$

For $p > 0$, we use the notation

$$\mathcal{E}^p(X, \omega) := \mathcal{E}_\chi(X, \omega), \text{ when } \chi(t) = -(-t)^p.$$

5.1.2 The (φ, ψ)-Capacity

In this subsection we always assume that $\varphi, \psi \in \text{PSH}(X, \omega)$ are such that $\varphi < \psi$ quasi everywhere on X. The (φ, ψ)-capacity of a Borel subset $E \subset X$ is defined by

$$\text{Cap}_{\varphi,\psi}(E) := \sup \left\{ \int_E \text{MA}(u) \;\middle|\; u \in \text{PSH}(X, \omega),\ \varphi \leq u \leq \psi \right\}.$$

When $\varphi \equiv \psi - 1$, to simplify the notation we simply denote

$$\text{Cap}_\psi := \text{Cap}_{\psi-1,\psi}.$$

If moreover $\psi \equiv C$ is constant we recover the Monge-Ampère capacity introduced in [BT82], [Koł03], [GZ05]. The following properties of the (φ, ψ)-Capacity follow straightforward from the definition.

Proposition 5.1.4. *(i) If $E_1 \subset E_2 \subset X$ then* $\text{Cap}_{\varphi,\psi}(E_1) \leq \text{Cap}_{\varphi,\psi}(E_2)$.
(ii) If $E_1, E_2, \cdots$ are Borel subsets of X then

$$\text{Cap}_{\varphi,\psi}\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{+\infty} \text{Cap}_{\varphi,\psi}(E_j).$$

(iii) If $E_1 \subset E_2 \subset \cdots$ are Borel subsets of X then

$$\text{Cap}_{\varphi,\psi}\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to+\infty} \text{Cap}_{\varphi,\psi}(E_j).$$

The *outer (φ, ψ)-capacity* of E is defined by

$$\text{Cap}^*_{\varphi,\psi}(E) := \inf \left\{ \text{Cap}_{\varphi,\psi}(U) \;\middle|\; U \text{ is an open subset of } X,\ E \subset U \right\}.$$

We say that the (φ, ψ)-capacity *characterizes pluripolar sets* on X if for any subset $E \subset X$, the following holds

$$\text{Cap}^*_{\varphi,\psi}(E) = 0 \Longleftrightarrow \text{ E is a pluripolar subset of } X.$$

Definition 5.1.5. If $E \subset X$ is a Borel subset we set

$$h_{\varphi,\psi,E} := \sup \{ u \in \text{PSH}(X, \omega),\ u \leq \varphi \text{ quasi everywhere on } E, u \leq \psi \text{ on } X \},$$

where "quasi everywhere" means outside a pluripolar set. The upper semicontinuous regularization of $h_{\varphi,\psi,E}$ is called the relative (φ, ψ)-extremal function of E.

Proposition 5.1.6. *Let $E \subset X$.*

(i) The function $h^*_{\varphi,\psi,E}$ is ω-psh. It satisfies $\varphi \leq h^*_{\varphi,\psi,E} \leq \psi$ on X and $h^*_{\varphi,\psi,E} = \varphi$ quasi everywhere on E.

(ii) If $P \subset E$ is pluripolar, then $h^*_{\varphi,\psi,E\setminus P} \equiv h^*_{\varphi,\psi,E}$; in particular $h^*_{\varphi,\psi,P} \equiv \psi$.

(iii) If (E_j) are subsets of X increasing towards $E \subset X$, then h^*_{φ,ψ,E_j} decreases towards $h^*_{\varphi,\psi,E}$.

(iv) If $h^*_{\varphi,\psi,E} \equiv \psi$ then E is pluripolar.

Proof. The statement (i) is a standard consequence of Bedford-Taylor's work [BT82]. Set $E_1 := E \setminus P$, and denote by $h = h^*_{\varphi,\psi,E}$, $h_1 = h^*_{\varphi,\psi,E_1}$ the corresponding (φ,ψ)-extremal functions of E, E_1. Since $E_1 \subset E$ it is clear that $h_1 \geq h$. On the other hand $h_1 = \varphi$ quasi everywhere on E_1 hence on E. This yields $h_1 \leq h$ whence equality.

Let us prove (iii). Since (E_j) is increasing, $h_j := h^*_{\varphi,\psi,E_j}$ is decreasing toward $h \in \mathrm{PSH}(X,\omega)$. It is clear that $h \geq h^*_{\varphi,\psi,E}$. By definition, for each $j \in \mathbb{N}$, $h_j = \varphi$ quasi everywhere on E_j. It then follows that $h = \varphi$ quasi everywhere on E. We then infer that $h \leq h^*_{\varphi,\psi,E}$, hence the equality.

To prove (iv) assume that $h^*_{\varphi,\psi,E} \equiv \psi$. By definition of $h := h^*_{\varphi,\psi,E}$ and by Choquet's lemma we can find an increasing sequence (u_j) such that $u_j = \varphi$ on E and $(\lim_{j\to+\infty} u_j)^* = h$. Note that

$$E \subset \left\{ \left(\limsup_{j\to+\infty} u_j \right) < \left(\limsup_{j\to+\infty} u_j \right)^* \right\},$$

modulo a pluripolar set. The latter is also pluripolar, hence E is pluripolar. □

Theorem 5.1.7. *If $\varphi \in \mathcal{E}(X,\omega)$ and $E \subset X$ is pluripolar then* $\mathrm{Cap}^*_{\varphi,\psi}(E) = 0$.

Proof. Assume that $\varphi \in \mathcal{E}(X,\omega)$ and fix a pluripolar set $E \subset X$. By translating ψ and φ by a constant we can assume that $\psi \leq 0$. It follows from [GZ07, Proposition 2.2] that $\varphi \in \mathcal{E}_\chi(X,\omega)$ for some convex increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$. We can find $u \in \mathcal{E}_\chi(X,\omega), u \leq 0$ such that $E \subset \{u = -\infty\}$. We claim that

$$\mathrm{Cap}_{\varphi,\psi}(\{u < -t\}) \leq \frac{-2}{\chi(-t)} \left(E_\chi(u) + 2^n E_\chi(\varphi) \right), \ \forall t > 0.$$

Indeed, let $v \in PSH(X,\omega)$ such that $\varphi \leq v \leq \psi$. We obtain immediately that

$$\int_{\{u<-t\}} \mathrm{MA}\,(v) \leq \frac{1}{-\chi(-t)} \int_{\{u<-t\}} (-\chi \circ u) \mathrm{MA}\,(v).$$

From this and [GZ07, Proposition 2.5] we get

$$\int_{\{u<-t\}} MA(v) \leq \frac{-2}{\chi(-t)} \left(E_\chi(u) + E_\chi(v)\right).$$

This coupled with the *fundamental inequality* in [GZ07, Lemma 2.3] yield the claim. Since for any $t > 0$, $E \subset \{u < -t\}$ we obtain

$$\mathrm{Cap}^*_{\varphi,\psi}(E) \leq \mathrm{Cap}_{\varphi,\psi}(u < -t) \to 0 \quad \text{as} \quad t \to +\infty.$$

□

From now on we fix φ, ψ two functions in $\mathcal{E}(X, \omega)$ such that $\varphi < \psi$ quasi everywhere on X.

Given any $u \in \mathrm{PSH}(X, \omega)$ such that $u \leq 0$, it follows from [GZ07, Example 2.14] (see also the Main Theorem in [CGZ08]) that $u_p := -(-u)^p$ belongs to $\mathcal{E}(X, \omega)$ for any $0 < p < 1$. The same arguments can be applied to get the following result:

Lemma 5.1.8. *Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be any measurable function. Assume that there exists $q > 0$ such that*

$$\sup_{t \leq -1} |\chi(t)|(-t)^{-q} = C < +\infty.$$

Then for any $u \in \mathrm{PSH}(X, \omega)$ such that $u \leq -1$ and any $0 < p < \frac{1}{q+1}$ we have

$$\int_X |\chi \circ u_p| \mathrm{MA}\,(u_p) \leq A,$$

where $u_p := -(-u)^p$ and A is a positive constant depending only on C, p, q.

Proof. In the proof we use A to denote various positive constants which are under control. By considering $u^j := \max(u, -j)$, the canonical approximants of u, and letting $j \to +\infty$ it suffices to treat the case when u is bounded. We compute

$$\omega + dd^c u_p = \omega + p(1-p)(-u)^{p-2} du \wedge d^c u + p(-u)^{p-1} dd^c u.$$

We thus get

$$0 \leq \omega + dd^c u_p \leq (-u)^{p-1}(\omega + dd^c u) + \omega + (-u)^{p-2} du \wedge d^c u.$$

We need to verify the following bounds:

$$\int_X |\chi \circ u_p|(-u)^{p-1}(\omega + dd^c u)^k \wedge \omega^{n-k} \leq A$$

and

$$\int_X |\chi \circ u_p|(-u)^{p-2} du \wedge d^c u \wedge (\omega + dd^c u)^k \wedge \omega^{n-k-1} \leq A,$$

where $k = 0, 1, ..., n$. Let us consider the first one. By assumption we have

$$|\chi \circ u_p|(-u_p)^{-q} \leq C.$$

To bound the first term, it thus suffices to get a bound for

$$\int_X (-u)^{p-1+pq}(\omega + dd^c u)^k \wedge \omega^{n-k},$$

which is easy since $p + pq - 1 < 0$. For the second one it suffices get a bound for

$$\int_X (-u)^{p-2+pq} du \wedge d^c u \wedge (\omega + dd^c u)^k \wedge \omega^{n-k-1},$$

which follows easily by the same reason and by integration by parts.

□

We know from Theorem 5.1.7 that $\mathrm{Cap}_{\varphi,\psi}$ vanishes on pluripolar subsets of X. This suggests that $\mathrm{Cap}_{\varphi,\psi}$ is dominated by $F(\mathrm{Cap}_\omega)$, where F is some positive function vanishing at 0. The following result gives an explicit formula of F.

Theorem 5.1.9. *Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex increasing function and $\varphi \in \mathcal{E}_\chi(X, \omega)$. Let $q > 0$ be a positive real number such that*

$$\sup_{t \leq -1} |\chi(t)|(-t)^{-q} < +\infty. \tag{5.1.1}$$

Then for any $p < \frac{1}{1+q}$ there exists $C > 0$ depending on p, q, χ, φ such that

$$\mathrm{Cap}_{\varphi,0}(K) \leq \frac{C}{\left|\chi\left(-\mathrm{Cap}_\omega(K)^{\frac{-p}{n}}\right)\right|}, \ \forall K \subset X.$$

As a concrete example, when $\varphi \in \mathcal{E}^q(X, \omega)$ for some $q > 0$ and $p < 1/(1+q)$, then we can take $F(s) := s^{\frac{pq}{n}}$ for $s > 0$, getting

$$\mathrm{Cap}_{\varphi,0}(K) \leq C\,\mathrm{Cap}_\omega(K)^{\frac{pq}{n}}.$$

Proof. Fix $p > 0$ such that $p(q + 1) < 1$. Let V_K be the extremal ω-plurisubharmonic function of K:

$$V_K := \sup\{u \mid u \in \mathrm{PSH}(X, \omega),\ u \leq 0 \text{ on K}\},$$

and $M_K := \sup_X V_K^*$. It follows from (5.1.1) and Lemma 5.1.8 that the function

$$u = -(-V_K^* + M_K + 1)^p$$

belongs to $\mathcal{E}_\chi(X, \omega)$. Fix $h \in \mathrm{PSH}(X, \omega)$ be such that $\varphi \leq h \leq 0$. It follows from Lemma 5.1.10 below that

$$\int_X |\chi \circ u| \mathrm{MA}\,(h) \leq C_1,$$

where $C_1 > 0$ only depends on χ, p, q and φ. Therefore, using the fact that $V_K^* \equiv 0$ quasi everywhere on K we get

$$\int_K \mathrm{MA}\,(h) \leq \int_X \frac{|\chi \circ u|}{|\chi(-M_K^p)|}\omega_h^n \leq \frac{C_1}{|\chi(-M_K^p)|}.$$

It follows from [GZ05] that $M_K \geq C_2 \mathrm{Cap}(K)^{-1/n}$. This coupled with the above yield the result. $\square$

Lemma 5.1.10. *Assume that χ, p, q and φ are as in Theorem 5.1.9. Then there exists $C > 0$ depending on χ, p, q, φ such that*

$$\int_X |\chi(-(-u)^p)|\mathrm{MA}\,(v) \leq C,\ \forall u, v \in \mathrm{PSH}(X,\omega),\ \sup_X u = -1,\ \varphi \leq v \leq 0.$$

Proof. We argue by contradiction, assuming that there are two sequences (u_j), (v_j) of functions in $\mathrm{PSH}(X,\omega)$ such that $\sup_X u_j = -1$, $\varphi \leq v_j \leq 0$, and

$$\int_X |\chi(-(-u_j)^p)|\mathrm{MA}\,(v_j) \geq 2^{(n+2)j},\ \forall j \in \mathbb{N}.$$

Set

$$u := \sum_{j=1}^{+\infty} 2^{-j} u_j,\ v = \sum_{j=1}^{+\infty} 2^{-j} v_j.$$

Then $u \in \mathrm{PSH}(X,\omega)$, $u \leq -1$. Moreover, it follows from Lemma 5.1.8 that

$$u_p := -(-u)^p \in \mathcal{E}_\chi(X,\omega).$$

We also have $\varphi \leq v \leq 0$, in particular $v \in \mathcal{E}_\chi(X,\omega)$. But

$$\int_X |\chi \circ u_p|\mathrm{MA}\,(v) \geq \sum_{j=1}^{+\infty} 2^j = +\infty,$$

which contradicts [GZ07, Proposition 2.5]. $\square$

Proposition 5.1.11. *Let E be a Borel subset of X and set $h_E := h^*_{\varphi,\psi,E}$ the relative (φ,ψ)-extremal function of E. Then*

$$\mathrm{MA}\,(h_E) \equiv 0 \text{ on } \{h_E < \psi\} \setminus \bar{E}.$$

Proof. We first assume that ψ is continuous on X. Set $h := h_E$ and let $x_0 \in X \setminus \bar{E}$ be such that $(h-\psi)(x_0) < 0$. Let $B := B(x_0, r) \subset X \setminus \bar{E}$ be a small ball around x_0 such that $\sup_{\bar{B}}(h-\psi)(x) = -2\delta < 0$. Let ρ be a local potential of ω in B. Shrinking B a little bit we can assume that $\sup_{\bar{B}} |\rho| < \delta$ and $\mathrm{osc}_{\bar{B}}\psi < \delta/2$. By definition of h and by Choquet's lemma we can find an increasing sequence $(u_j)_j \subset \mathcal{E}(X,\omega)$ such that $u_j = \varphi$ quasi everywhere on E, $u_j \leq \psi$ on X, and $(\lim_j u_j)^* = h$. For each $j, k \in \mathbb{N}$, we solve the

Dirichlet problem to find $v_j^k \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ such that $\mathrm{MA}\,(v_j^k) = 0$ in B and $v_j^k \equiv \max(u_j, -k)$ on $X \setminus B$. Since

$$\rho + v_j^k \leq \rho + h \leq -\delta + \psi \leq \sup_{\bar{B}} \psi - \delta$$

on ∂B, we deduce from the maximum principle that $v_j^k \leq \inf_{\bar{B}} \psi - \delta/2 - \rho \leq \psi$ on B. Furthermore, taking k big enough such that $\psi \geq -k$, we can conclude that $v_j^k \leq \psi$ on X. For $j \in \mathbb{N}$ fixed, by the comparison principle $(v_j^k)_k$ decreases to $v_j \in \mathcal{E}(X,\omega)$. Then $u_j \leq v_j \leq h$ since $v_j = u_j = \varphi$ on E and $v_j \leq \psi$ on X. It follows from [GZ07] that the sequence of Monge-Ampère measures $MA(v_j^k)$ converges weakly to $MA(v_j)$. Thus $MA(v_j)(B) = 0$. On the other hand, v_j increases almost everywhere to h and these functions belong to $\mathcal{E}(X,\omega)$. The same arguments as in [GZ07, Theorem 2.6] show that $MA(v_j)$ converges weakly to $MA(h)$. We infer that $MA(h)(B) = 0$.

It remains to remove the continuity hypothesis on ψ. Let (ψ_j) be a sequence of continuous functions in $\mathrm{PSH}(X,\omega)$ decreasing to ψ on X. Let $h_j := h^*_{\varphi,\psi_j,E}$ be the relative (φ,ψ_j)-extremal function of K. Then h_j decreases to h, hence $\mathrm{MA}\,(h_j)$ converges weakly to $\mathrm{MA}\,(h)$. Denote by $V := \{h < \psi\} \setminus \bar{E}$. Now, fix $\varepsilon > 0$ and U an open subset of X such that

$$\mathrm{Cap}_\omega\,[(U \setminus V) \cup (V \setminus U)] \leq \varepsilon.$$

From the first step we know that $\mathrm{MA}\,(h_j)$ vanishes on V. Thus

$$\begin{aligned}
\int_V \mathrm{MA}\,(h) &\leq \int_U \mathrm{MA}\,(h) + F(\varepsilon) \\
&\leq \liminf_{j \to +\infty} \int_U \mathrm{MA}\,(h_j) + F(\varepsilon) \\
&\leq \liminf_{j \to +\infty} \int_V \mathrm{MA}\,(h_j) + 2F(\varepsilon) \\
&= 2F(\varepsilon),
\end{aligned}$$

It suffices now to let $\varepsilon \to 0$ since $\lim_{\varepsilon \to 0} F(\varepsilon) = 0$ thanks to Theorem 5.1.9. □

Lemma 5.1.12. *Let $E \subset X$ be a Borel subset and $h_E := h^*_{\varphi,\psi,E}$ be its relative (φ,ψ)-extremal function. Then we have*

$$\mathrm{Cap}_{\varphi,\psi}(E) \leq \int_{\{h_E < \psi\}} \mathrm{MA}\,(h_E).$$

Proof. Observe first that the (φ,ψ)-capacity can be equivalently defined by

$$\mathrm{Cap}_{\varphi,\psi}(E) := \sup\left\{ \int_E \mathrm{MA}\,(u) \mid u \in \mathrm{PSH}(X,\omega),\ \varphi < u \leq \psi \right\}.$$

For simplicity, set $h := h_E$. Now take any $u \in \mathrm{PSH}(X,\omega)$ such that $\varphi < u \leq \psi$. Then

$$E \subset \{h < u\} \subset \{h < \psi\},$$

where the first inclusion holds modulo a pluripolar set. The comparison principle for functions in $\mathcal{E}(X,\omega)$ (see [GZ07]) yields

$$\int_E MA(u) \leq \int_{\{h<u\}} MA(u) \leq \int_{\{h<u\}} MA(h) \leq \int_{\{h<\psi\}} MA(h).$$

By taking the supremum over all candidates u, we get the result. □

The following result says that the inequality in Lemma 5.1.12 is an equality if E is a compact or open subset of X.

Theorem 5.1.13. *Let E be an open (or compact) subset of X and let $h_E := h^*_{\varphi,\psi,E}$ be the (φ,ψ)-extremal function of E. The (φ,ψ)-capacity of E is given by*

$$\mathrm{Cap}_{\varphi,\psi}(E) = \int_{\{h_E<\psi\}} MA(h_E).$$

Proof. From Lemma 5.1.12 above we get one inequality. We now prove the opposite one. Set $h := h_E$. Assume first that E is a compact subset of X. Let (ψ_j) be a sequence of continuous ω-psh functions decreasing to ψ. Denote by $h_j := h^*_{\varphi,\psi_j,E}$. It is easy to check that h_j decreases to h and that $\mathrm{Cap}_{\varphi,\psi_j}(E)$ decreases to $\mathrm{Cap}_{\varphi,\psi}(E)$. Since h_j is a candidate defining the (φ,ψ_j)-capacity of E, it follows from Proposition 5.1.11 and Lemma 5.1.12 that

$$\mathrm{Cap}_{\varphi,\psi_j}(E) = \int_{\{h_j<\psi_j\}} MA(h_j) = \int_E MA(h_j). \tag{5.1.2}$$

Fix $j_0 \in \mathbb{N}$. Since $h_j \leq h_{j_0}$ and $\psi \leq \psi_j$, for any $j > j_0$

$$\int_{\{h_j<\psi_j\}} MA(h_j) \geq \int_{\{h_{j_0}<\psi\}} MA(h_j).$$

Fix $\varepsilon > 0$ and replacing ψ by a continuous function $\tilde{\psi}$ such that $\mathrm{Cap}_\omega(\{\tilde{\psi} \neq \psi\}) < \varepsilon$. Arguing as in the proof of Proposition 5.1.11 we get

$$\liminf_{j\to+\infty} \int_{\{h_{j_0}<\psi\}} MA(h_j) \geq \int_{\{h_{j_0}<\psi\}} MA(h).$$

Taking the limit for $j \to +\infty$ in (5.1.2) we get

$$\mathrm{Cap}_{\varphi,\psi}(E) \geq \int_{\{h<\psi\}} MA(h).$$

We now assume that $E \subset X$ is an open set. Let (K_j) be a sequence of compact subsets increasing to E. Then clearly $h_j := h^*_{\varphi,\psi,K_j} \searrow h$ and

$\mathrm{Cap}_{\varphi,\psi}(K_j) \nearrow \mathrm{Cap}_{\varphi,\psi}(E)$. We have already proved that $\mathrm{Cap}_{\varphi,\psi}(K_j) \geq \int_{\{h_j<\psi\}} MA(h_j)$. For each fixed $k \in \mathbb{N}$, we have

$$\liminf_{j\to+\infty} \int_{\{h_j<\psi\}} MA(h_j) \geq \liminf_{j\to+\infty} \int_{\{h_k<\psi\}} MA(h_j) \geq \int_{\{h_k<\psi\}} MA(h).$$

Then letting $k \to +\infty$ and using the first part of the proof we get

$$\liminf_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(K_j) \geq \int_{\{h<\psi\}} MA(h).$$

On the other hand, it is clear that $\lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(K_j) = \mathrm{Cap}_{\varphi,\psi}(E)$, and hence

$$\mathrm{Cap}_{\varphi,\psi}(E) \geq \int_{\{h<\psi\}} MA(h).$$

□

Now we want to give a formula for the outer (φ,ψ)-capacity. Assume that E is a Borel subset of X. We introduce an auxiliary function

$$\phi := \phi_{\varphi,\psi,E} = \begin{cases} \frac{-h^*_{\varphi,\psi,E}+\psi}{-\varphi+\psi} & \text{if } \varphi > -\infty \\ 0 & \text{if } \varphi = -\infty \end{cases}. \tag{5.1.3}$$

Observe that ϕ is a quasicontinuous function, $0 \leq \phi \leq 1$ and $\phi = 1$ quasi everywhere on E.

Theorem 5.1.14. *Let $E \subset X$ be a Borel subset and denote by $h_E := h^*_{\varphi,\psi,E}$ the (φ,ψ)-extremal function of E. Then*

$$\mathrm{Cap}^*_{\varphi,\psi}(E) = \int_{\{h_E<\psi\}} \mathrm{MA}\,(h_E) = \int_X \left(\frac{-h_E+\psi}{-\varphi+\psi}\right) \mathrm{MA}\,(h_E).$$

To prove Theorem 5.1.14 we need the following results.

Lemma 5.1.15. *Let (u_j) be a bounded monotone sequence of quasi-continuous functions converging to u. Let χ be a convex weight and $\{\varphi_j\} \subset \mathcal{E}_\chi(X,\omega)$ be a monotone sequence converging to $\varphi \in \mathcal{E}_\chi(X,\omega)$. Then*

$$\int_X u_j \,\mathrm{MA}\,(\varphi_j) \xrightarrow[j\to+\infty]{} \int_X u \,\mathrm{MA}\,(\varphi).$$

Proof. Fix $\varepsilon > 0$. Let U be an open subset of X with $\mathrm{Cap}_\omega(U) < \varepsilon$ and v_j, v be continuous functions on X such that $v_j \equiv u_j$ and $v \equiv u$ on $K := X \setminus U$. By Theorem 5.1.9 (and by letting $\varepsilon \to 0$) it suffices to prove that

$$\int_X v_j \,\mathrm{MA}\,(\varphi_j) \xrightarrow[j\to+\infty]{} \int_X v \,\mathrm{MA}\,(\varphi)\,.$$

From Dini's theorem v_j converges uniformly to v on K. Thus, using again Theorem 5.1.9, the problem reduces to proving that

$$\int_X v\,\mathrm{MA}\,(\varphi_j) \xrightarrow[j\to+\infty]{} \int_X v\,\mathrm{MA}\,(\varphi)\,.$$

But the latter obviously follows since v is continuous on X. The proof is thus complete. $\square$

Proposition 5.1.16. *Let E be a compact or open subset of X and let $h_E := h^*_{\varphi,\psi,E}$ denote the (φ,ψ)-extremal function of E. Then*

$$\mathrm{Cap}_{\varphi,\psi}(E) = \int_{\{h_E<\psi\}} \mathrm{MA}\,(h_E) = \int_X \left(\frac{-h_E+\psi}{-\varphi+\psi}\right) \mathrm{MA}\,(h_E).$$

Proof. The first equality has been proved in Theorem 5.1.13. Set $h := h_E$ and $\phi := \phi_{\varphi,\psi,E} = \frac{-h_E+\psi}{-\varphi+\psi}$. Observe that $\{h<\psi\} = \{\phi>0\}$ modulo a pluripolar set and $\phi \le 1$. Thus

$$\int_{\{h<\psi\}} \mathrm{MA}\,(h) \ge \int_X \phi\,\mathrm{MA}\,(h).$$

Assume that E is compact. By Proposition 5.1.11 and Theorem 5.1.13 we have

$$\mathrm{Cap}_{\varphi,\psi}(E) = \int_E \mathrm{MA}\,(h).$$

Since $\phi = 1$ quasi everywhere on E we obtain

$$\int_E \mathrm{MA}\,(h) \le \int_X \phi\,\mathrm{MA}\,(h).$$

We assume now that $E \subset X$ is an open subset. Let (K_j) be a sequence of compact subsets increasing to E. Then

$$\mathrm{Cap}_{\varphi,\psi}(E) = \lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(K_j) = \lim_{j\to+\infty} \int_X \phi_j\,\mathrm{MA}\,(h_j),$$

where $h_j := h^*_{\varphi,\psi,K_j}$ and $\phi_j := \phi_{\varphi,\psi,K_j}$ is defined by (5.1.3). Since ϕ_j is quasicontinuous for any j and $\phi_j \searrow \phi$, the conclusion follows from Lemma 5.1.15. $\square$

Lemma 5.1.17. *Let u, v be ω-plurisubharmonic functions. Let $G \subset X$ be an open subset. Set $E = \{u<v\} \cap G$ and $h_E := h^*_{\varphi,\psi,E}$. Then*

$$\mathrm{Cap}^*_{\varphi,\psi}(E) = \mathrm{Cap}_{\varphi,\psi}(E) = \int_{\{h_E<\psi\}} \mathrm{MA}\,(h_E) = \int_X \left(\frac{-h_E+\psi}{-\varphi+\psi}\right) \mathrm{MA}\,(h_E).$$

Proof. We start showing the first identity. First, just by definition $\mathrm{Cap}^*_{\varphi,\psi}(E) \geq \mathrm{Cap}_{\varphi,\psi}(E)$. Fix $\varepsilon > 0$. There exists a function $\tilde{v} \in \mathcal{C}(X)$ such that

$$\mathrm{Cap}_\omega(\{\tilde{v} \neq v\}) < \varepsilon.$$

Clearly $E \subset (\{u < \tilde{v}\} \cap G) \cup \{\tilde{v} \neq v\}$ and so, applying Theorem 5.1.9 we get

$$\begin{aligned} \mathrm{Cap}^*_{\varphi,\psi}(E) &\leq \mathrm{Cap}_{\varphi,\psi}(\{u < \tilde{v}\} \cap G) + F(\varepsilon) \\ &\leq \mathrm{Cap}_{\varphi,\psi}(E) + 2F(\varepsilon), \end{aligned}$$

where $F(\varepsilon) \to 0$ as $\varepsilon \to 0$. Taking the limit as $\varepsilon \to 0$ we arrive at the first conclusion.

Let now $\{K_j\}$ be a sequence of compact sets increasing to G and $\{u_j\}$ be a sequence of continuous functions decreasing to u. Then $E_j = \{u_j + 1/j \leq v\} \cap K_j$ is compact and $E_j \nearrow E$. Set

$$h := h_{\varphi,\psi,E},\ \phi := \frac{-h_E + \psi}{-\varphi + \psi},\ h_j := h^*_{\varphi,\psi,E_j},\ \phi_j := \frac{-h_{E_j} + \psi}{-\varphi + \psi}.$$

Observe that $h_j \searrow h$ and $\phi_j \searrow \phi$. By Proposition 5.1.16 and Lemma 5.1.15 we have

$$\begin{aligned} \mathrm{Cap}_{\varphi,\psi}(E) &= \lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(E_j) \\ &= \lim_{j\to+\infty} \int_X \phi_j \,\mathrm{MA}\,(h_j) \\ &= \int_X \phi \,\mathrm{MA}\,(h) \leq \int_{\{h<\psi\}} \mathrm{MA}\,(h). \end{aligned}$$

Furthermore, for each fixed $k \in \mathbb{N}$, using Theorem 5.1.9 we can argue as above to get

$$\liminf_{j\to+\infty} \int_{\{h_j<\psi\}} \mathrm{MA}\,(h_j) \geq \liminf_{j\to+\infty} \int_{\{h_k<\psi\}} \mathrm{MA}\,(h_j) \geq \int_{\{h_k<\psi\}} \mathrm{MA}\,(h).$$

Letting $k \to +\infty$ and using Proposition 5.1.16 again we get

$$\mathrm{Cap}_{\varphi,\psi}(E) \geq \int_{\{h<\psi\}} \mathrm{MA}\,(h),$$

which completes the proof. □

We are now ready to prove Theorem 5.1.14.

Proof. As usual, for simplicity, set $h := h_E$. By definition of the outer capacity there is a sequence (O_j) of open sets decreasing to E such that $\mathrm{Cap}^*_{\varphi,\psi}(E) = \lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(O_j)$. Furthermore by Choquet's lemma there exists a sequence (u_j) of ω-psh functions such that $u_j \equiv \varphi$ quasi everywhere on E, $u_j \leq \psi$ on X and $u_j \nearrow h$. Since $\mathrm{Cap}^*_{\varphi,\psi}$ vanishes on pluripolar sets (see Theorem 5.1.7) we can assume that $u_j \equiv \varphi$ on E. For any j, we set $E_j = O_j \cap \{u_j < \varphi + 1/j\}$ and $h_j := h^*_{\varphi,\psi,E_j}$. Then (E_j) is a decreasing sequence of open subsets such that $E \subset E_j \subset O_j$ and $u_j - 1/j \leq h_j \leq h$, thus $h_j \nearrow h$. Clearly $\mathrm{Cap}^*_{\varphi,\psi}(E) = \lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(E_j)$. By Lemma 5.1.17 and Lemma 5.1.15 we get

$$\lim_{j\to+\infty} \mathrm{Cap}^*_{\varphi,\psi}(E_j) = \lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(E_j) = \lim_{j\to+\infty} \int_X \phi_j \, \mathrm{MA}\,(h_j) = \int_X \phi \, \mathrm{MA}\,(h),$$

where $\phi_j := \phi_{\varphi,\psi,E_j}$ is defined by (5.1.3). □

Corollary 5.1.18. *Let $K \subset X$ be a compact set and (K_j) a sequence of compact subsets decreasing to K. Then*

(i) $\mathrm{Cap}^*_{\varphi,\psi}(K) = \mathrm{Cap}_{\varphi,\psi}(K) = \lim_{j\to+\infty} \mathrm{Cap}_{\varphi,\psi}(K_j)$,

(ii) $h^*_{\varphi,\psi,K_j} \nearrow h^*_{\varphi,\psi,K}$.

Proof. The first equality in statement (i) comes straightforward from Theorem 5.1.13 and Theorem 5.1.14. The second one follows from (ii) and Theorem 5.1.14. It remains to prove (ii). Since (K_j) decreases to K, $h_j := h^*_{\varphi,\psi,K_j}$ increases to some $h_\infty \in \mathcal{E}(X,\omega)$. Clearly $h_\infty \leq h$. Thus we need to prove that $h_\infty \geq h$. Since $\{h_\infty < h\} \subset \{h_\infty < \psi\} \setminus K$ modulo a pluripolar set,

$$\int_{\{h_\infty < h\}} \mathrm{MA}\,(h_\infty) \leq \int_{\{h_\infty<\psi\}\setminus K} \mathrm{MA}\,(h_\infty).$$

From Proposition 5.1.11 we know that

$$\int_{\{h_j<\psi\}\setminus K_j} \mathrm{MA}\,(h_j) = 0, \; \forall j \in \mathbb{N}.$$

Fix $\varepsilon > 0$ and let $\psi_\varepsilon \in \mathcal{C}(X)$ such that $\mathrm{Cap}_\omega(\{\psi_\varepsilon \neq \psi\}) < \varepsilon$. Then for each fixed $k \in \mathbb{N}$, we have

$$\begin{aligned}
\int_{\{h_\infty<\psi\}\setminus K_k} \mathrm{MA}\,(h_\infty) &\leq \int_{\{h_\infty<\psi_\varepsilon\}\setminus K_k} \mathrm{MA}\,(h_\infty) + F(\varepsilon) \\
&\leq \liminf_{j\to+\infty} \int_{\{h_\infty<\psi_\varepsilon\}\setminus K_k} \mathrm{MA}\,(h_j) + F(\varepsilon) \\
&\leq \liminf_{j\to+\infty} \int_{\{h_\infty<\psi\}\setminus K_k} \mathrm{MA}\,(h_j) + 2F(\varepsilon) \\
&\leq \liminf_{j\to+\infty} \int_{\{h_j<\psi\}\setminus K_k} \mathrm{MA}\,(h_j) + 2F(\varepsilon) \\
&= 2F(\varepsilon),
\end{aligned}$$

where $F(\varepsilon) \to 0$ as $\varepsilon \to 0$ thanks to Theorem 5.1.9. Thus, letting $\varepsilon \to 0$ then $k \to +\infty$ and using the domination principle below (Proposition 5.2.1) we can conclude that $h_\infty \geq h$. □

5.1.3 Proof of Theorem A

Let us briefly resume the proof of Theorem A. Statements (i) and (ii) have been proved in Theorem 5.1.14 and Theorem 5.1.9 respectively. One direction of the last staement has been proved in Theorem 5.1.7. Now, if E is a Borel subset of X such that $\mathrm{Cap}^*_{\varphi,\psi}(E) = 0$ then it follows from Theorem 5.1.14 that

$$\int_{\{h^*_{\varphi,\psi,E}<\psi\}} \mathrm{MA}\left(h^*_{\varphi,\psi,E}\right) = 0.$$

We then can apply the domination principle (see [BL12] or Proposition 5.2.1 below for a proof) to conclude.

5.2 Another proof of the Domination Principle

The following domination principle was proved by Dinew using his uniqueness result [Din09], [BL12]. As an application of the (φ, ψ)-Capacity we propose here an alternative proof.

Proposition 5.2.1. *If $u, v \in \mathcal{E}(X, \omega)$ such that $u \leq v$ $MA(v)$-almost everywhere then $u \leq v$ on X.*

Proof. We first claim that for every $\varphi \in \mathcal{E}(X, \omega)$ such that $0 \leq \varphi - u \leq C$ for some constant $C > 0$ and for any $s > 0$ one has

$$\int_{\{v<u-s\}} MA(\varphi) = 0.$$

Indeed, fix $s > 0$ and let φ be such a function. Let $C > 0$ be a constant such that $\varphi - u \leq C$ on X. Choose $\delta \in (0, 1)$ such that $\delta C < s$. Now, by using the comparison principle and the fact that $0 \leq \varphi - u \leq C$ we get

$$\begin{aligned}
\delta^n \int_{\{v<u-s\}} MA(\varphi) &= \int_{\{v<u-s\}} (\delta\omega + dd^c \delta\varphi)^n \\
&\leq \int_{\{v<\delta\varphi+(1-\delta)u-s\}} MA\left(\delta\varphi + (1-\delta)u\right) \\
&\leq \int_{\{v<\delta\varphi+(1-\delta)u-s\}} MA(v) \\
&\leq \int_{\{v<u\}} MA(v) = 0.
\end{aligned}$$

Thus, the claim is proved. Now for each $t > 0$ let h_t denote the $(u, 0)$-extremal function of the open set $G_t := \{u < -t\}$. It is clear that for every $t > 0$, $h_t \in \mathcal{E}(X, \omega)$ and $\sup_X(h_t - u) < +\infty$. The previous step yields

$$\int_{\{v<u-s\}} MA(h_t) = 0, \ \forall s > 0.$$

Fix $\varepsilon > 0$. Let $\tilde{u}$ be a continuous function on X such that $\mathrm{Cap}_\omega(\{u \neq \tilde{u}\}) < \varepsilon$. Since h_t increases to 0 (see Lemma 5.2.2 below), we infer that

$$\int_{\{v<\tilde{u}-s\}} \omega^n \leq \liminf_{t\to+\infty} \int_{\{v<u-s\}} \mathrm{MA}\,(h_t) + \mathrm{Cap}_{u,0}(\{u \neq \tilde{u}\}).$$

Repeating this argument we get

$$\int_{\{v<u-s\}} \omega^n \leq \varepsilon + \mathrm{Cap}_{u,0}(\{u \neq \tilde{u}\}).$$

Letting $\varepsilon \to 0$ and using Theorem 5.1.9 we get $\mathrm{Vol}(\{v < u - s\}) = 0$, for any $s > 0$ which implies that $u \leq v$ on X as desired. □

Lemma 5.2.2. *Let $v \in \mathrm{PSH}(X, \omega)$. For each $t > 0$, set $G_t := \{v < -t\}$. Denote by h_t the $(\varphi, 0)$-extremal function of G_t. Then h_t increases quasi everywhere on X to 0 when t increases to $+\infty$.*

Proof. We know that h_t increases quasi everywhere to $h \in \mathcal{E}(X, \omega)$ and that $h \leq 0$. By Theorem 5.1.7 (up to consider $-(-v)^p$ with $p \in (0, 1)$ instead of v), we get

$$\lim_{t\to+\infty} \mathrm{Cap}_{\varphi,0}(G_t) = 0.$$

It follows from Theorem 5.1.13 that for each $t > 0$,

$$\int_{\{h<0\}} MA(h_t) \leq \int_{\{h_t<0\}} MA(h_t) = \mathrm{Cap}_{\varphi,0}(G_t).$$

We thus get

$$\int_{\{h<0\}} MA(h) \leq \liminf_{t\to+\infty} \int_{\{h<0\}} MA(h_t) = 0.$$

The comparison principle yields $\mathrm{Vol}(\{h < 0\}) = 0$ which completes the proof. □

Remark 5.2.3. Lemma 5.2.2 is stated and proved in the case $\psi \equiv 0$. Observe that it also holds for any $\psi \in \mathcal{E}(X, \omega)$ such that $\varphi < \psi$. To see this we can follow the same arguments of above but for the final step where we get $\psi \leq h$ $\mathrm{MA}\,(h)$-almost everywhere. We then conclude using the domination principle.

5.3 Applications to Complex Monge-Ampère equations

In this section (in the same spirit of [DNL14a]) we prove Theorem B by using $\mathrm{Cap}_\psi := \mathrm{Cap}_{\psi-1,\psi}$. Let us recall the setting. Let X be a compact Kähler manifold of dimension n and let ω be a Kähler form on X. Let D be an arbitrary divisor on X. Consider the complex Monge-Ampère equations

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\omega^n,\ \lambda \in \mathbb{R}. \tag{5.3.1}$$

We say that f satisfies Condition $\mathcal{H}_f$ if

$$f = e^{\psi^+ - \psi^-},\ \ \psi^\pm \text{ are quasi psh functions on } X\ ,\ \psi^- \in L^\infty_{\mathrm{loc}}(X \setminus D).$$

We have already treated the case when $\lambda = 0$ in [DNL14a]. If $\lambda > 0$ and f is integrable then the same arguments can be applied. More precisely, $\mathcal{C}^0$-estimates follow from comparison principle while the $\mathcal{C}^2$ estimate follows exactly the same way as in [DNL14a].

The case when $\lambda < 0$ is known to be much more difficult. We need the following observation where we make use of the generalized capacity Cap_ψ:

Lemma 5.3.1. *Let $\varphi \in \mathcal{E}(X,\omega)$ be normalized by $\sup_X \varphi = 0$. Assume that there exist a positive constant A and $\psi \in \mathrm{PSH}(X, \omega/2)$ such that $\mathrm{MA}\,(\varphi) \leq e^{-A\psi}\omega^n$. Then there exists $C > 0$ depending only on $\int_X e^{-2A\varphi}\omega^n$ such that*

$$\varphi \geq \psi - C.$$

Observe that for all $A > 0$ and $\varphi \in \mathcal{E}(X,\omega)$, $e^{-A\varphi}\omega^n \in L^1(X)$ as follows from Skoda integrability theorem [Sko72], [Zer01], since functions in $\mathcal{E}(X,\omega)$ have zero Lelong number at all points [GZ07].

Proof. Set

$$H(t) = \left[\mathrm{Cap}_\psi(\{\varphi < \psi - t\})\right]^{1/n},\ t > 0.$$

Observe that $H(t)$ is right-continuous and $H(+\infty) = 0$ (see [DNL14a, Lemma 2.6]). It follows from [DNL14a, Lemma 2.7] that $\mathrm{Cap}_\omega \leq 2^n \mathrm{Cap}_\psi$. By a strong volume-capacity domination in [GZ05] we also have

$$\mathrm{vol}_\omega \leq \exp\left(\frac{-C_1}{\mathrm{Cap}_\omega^{1/n}}\right),$$

where C_1 depends only on (X,ω). Thus using [DNL14a, Proposition 2.8] and the assumption on the measure MA (φ), we get

$$\begin{aligned} s^n \mathrm{Cap}_\psi(\{\varphi<\psi-t-s\}) &\leq \int_{\{\varphi<\psi-t\}} \mathrm{MA}\,(\varphi) \\ &\leq \int_{\{\varphi<\psi-t\}} e^{-A\varphi} e^{A\psi} \mathrm{MA}\,(\varphi) \\ &\leq \left[\int_X e^{-2A\varphi}\omega^n\right]^{1/2} \left[\int_{\{\varphi<\psi-t\}} \omega^n\right]^{1/2} \\ &\leq C_2 \left[\mathrm{Cap}_\psi(\{\varphi<\psi-t\})\right]^2, \end{aligned}$$

where C_2 depends on $\int_X e^{-2A\varphi}\omega^n$. We then get

$$sH(t+s) \leq C_2^{1/n} H(t)^2, \ \forall t>0, \forall s \in [0,1].$$

Then by [EGZ09, Lemma 2.4] we get $\varphi \geq \psi - C_3$, where C_3 only depends on $\int_X e^{-2A\varphi}\omega^n$. □

Now, we are ready to prove Theorem B.

5.3.1 Proof of Theorem B

It suffices to treat the case when $\lambda = -1$. Since f satisfies Condition $\mathcal{H}_f$ we can write $\log f = \psi^+ - \psi^-$, where $\psi^\pm$ are qpsh functions on X, ψ^- is locally bounded on $X \setminus D$ and there exists a uniform constant $C>0$ such that

$$dd^c\psi^\pm \geq -C\omega, \ \sup_X \psi^+ \leq C.$$

We apply the smoothing kernel ρ_ε in Demailly regularization theorem [Dem92] to the functions φ and $\psi^\pm$. For ε small enough, we get

$$dd^c\rho_\varepsilon(\varphi+\psi^-) \geq -C_1\omega, \ \ dd^c\rho_\varepsilon(\psi^+) \geq -C_1\omega, \ \ \sup_X \rho_\varepsilon(\psi^+) \leq C_1,$$

where C_1 depends on C and the Lelong numbers of the currents $C\omega + dd^c\psi^\pm$. By the classical result of Yau [Yau78], for each ε, there exists a unique smooth ω-psh function ϕ_ε satisfying

$$\mathrm{MA}\,(\phi_\varepsilon) = e^{c_\varepsilon + \rho_\varepsilon(\psi^+) - \rho_\varepsilon(\varphi+\psi^-)}\omega^n = g_\varepsilon \omega^n, \ \ \sup_X \phi_\varepsilon = 0,$$

where c_ε is a normalization constant such that

$$\int_X g_\varepsilon \omega^n = \int_X e^{-\varphi} f \omega^n = \int_X \omega^n.$$

Since by Jensen's inequality $e^{\rho_\varepsilon(-\varphi+\log f)} \leq \rho_\varepsilon(e^{-\varphi+\log f})$ and $e^{\rho_\varepsilon(-\varphi+\log f)}$ converges point-wise to $e^{-\varphi}f$ on X, it follows from the general Lebesgue dominated convergence theorem that $e^{\rho_\varepsilon(-\varphi+\log f)}$ converges to $e^{-\varphi}f$ in $L^1(X)$ when $\varepsilon \downarrow 0$. This means that c_ε converges to zero when $\varepsilon \to 0$. It then follows from [DNL14a, Lemma 3.4] that ϕ_ε converges in $L^1(X)$ to $\varphi - \sup_X \varphi$. We now apply the $\mathcal{C}^2$ estimate in [DNL14a, Theorem 3.2] to get

$$n + \Delta\phi_\varepsilon \leq C_3 e^{-2\rho_\varepsilon(\varphi+\psi^-)} \leq C_4 e^{-2(\varphi+\psi^-)},$$

where C_3, C_4 are uniform constants (do not depend on ε). Now, we need to bound φ from below. By the assumption on f we have

$$\mathrm{MA}\,(\varphi) = e^{\psi^+ - (\varphi+\psi^-)}\omega^n \leq e^{-(\varphi+\psi^- - C)}\omega^n.$$

Consider $\psi := \frac{1}{2C+2}(\varphi + \psi^-)$. Since this function belongs to $\mathrm{PSH}(X, \omega/2)$ we can apply Lemma 5.3.1 to get

$$\varphi - \sup_X \varphi \geq \psi - C_5.$$

This gives $\varphi \geq C_6\psi^- - C_7$. Applying again this argument to ϕ_ε and noting that c_ε converges to 0, and hence under control, we get

$$\phi_\varepsilon \geq \rho_\varepsilon(\varphi + \psi^-) - C_8 \geq C_9\psi^- - C_{10}.$$

We can now conclude using the same arguments in [DNL14a, Section 3.3].

5.3.2 (Non) Existence of solutions

In the previous subsection, no regularity assumption on D has been done. We now discuss about the existence of solutions in concrete examples, assuming more information on D, f.

Let $D = \sum_{j=1}^N D_j$ be a simple normal crossing divisor on X. Reacall that "simple normal crossing" means that around each intersection point of k components $D_{j_1}, ..., D_{j_k}$ $(k \leq N)$, we can find complex coordinates $z_1, ..., z_n$ such that for each $l = 1, ..., k$ the hypersurface D_{j_l} is locally given by $z_l = 0$.

For each j, let L_j be the holomorphic line bundle defined by D_j. Let s_j be a holomorphic section of L_j defining D_j, i.e $D_j = \{s_j = 0\}$. We fix a hermitian metric h_j on L_j such that $|s_j| := |s_j|_{h_j} \leq 1/e$.

We assume that f has the following particular form:

$$f = \frac{h}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^{1+\alpha}},\ \alpha > 0, \tag{5.3.2}$$

where h is a bounded function: $0 < 1/B \leq h \leq B,\ B > 0$.
In this subsection we always assume that $\lambda < 0$.

Proposition 5.3.2. *Assume that f satisfies (5.3.2) with $0 < \alpha \leq 1$. Then there is no solution in $\mathcal{E}(X, \omega)$ to equation*

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\omega^n.$$

Proof. We can assume (up to normalization) that $\lambda = -1$. Then observe that if there exists $\varphi \in \mathcal{E}(X,\omega)$ such that $(\omega + dd^c\varphi)^n = e^{-\varphi}\mu$, where μ is a positive measure, then we can find $A > 0$ such that

$$\mu \leq A\,(\omega + dd^c u)^n\,,$$

where $u := e^{(\varphi - \sup_X \varphi)/n}$ is a bounded ω-psh function. Indeed, u is a ω-psh function and

$$\omega + dd^c u \geq \omega + \frac{u}{n} dd^c\varphi \geq \frac{u}{n}(\omega + dd^c\varphi) \geq 0.$$

This coupled with [DNL14a, Proposition 4.4 and 4.5] yields the conclusion. □

The above analysis shows that there is no solution if the density has singularities of Poincaré type or worse. We next show that when f is less singular than the Poincaré type density (i.e. $\alpha > 1$), equation (5.3.1) has a bounded solution provided $\lambda = -\varepsilon$ with $\varepsilon > 0$ very small. We say that f satisfies Condition $\mathcal{S}(B,\alpha)$ for some $B > 0$, $\alpha > 0$ if

$$f \leq \frac{B}{\prod_{j=1}^N |s_j|^2(-\log|s_j|)^{1+\alpha}}.$$

Theorem 5.3.3. *Assume that f satisfies Condition $\mathcal{S}(B,\alpha)$ with $\alpha > 1$. We also normalize f so that $\int_X f\omega^n = \int_X \omega^n$. Then for $\lambda = -\varepsilon$ with $\varepsilon > 0$ small enough depending only on C,α,ω, there exists a bounded solution φ to (5.3.1).*
The solution is automatically continuous on X. In particular, it is also smooth on $X \setminus D$ if f is smooth there.

Proof. The last statement follows easily from our previous analysis. Let us prove the existence. We use the Schauder Fixed Point Theorem. Let $C = C(2B,\alpha)$ be the constant in Lemma 5.3.4 below. Choose $\varepsilon > 0$ very small such that $e^{\varepsilon C} \leq 2$. Consider the following compact convex set in $L^1(X)$:

$$\mathcal{C} := \{u \in \mathrm{PSH}(X,\omega) \ \big|\ -C \leq u \leq 0\}.$$

Let $\psi \in \mathcal{C}$ and c_ψ be a constant such that

$$\int_X e^{-\varepsilon\psi + c_\psi} f\omega^n = \int_X \omega^n.$$

Since $-C \leq \psi \leq 0$, it is clear that $-C\varepsilon \leq c_\psi \leq 0$. Let φ be the unique bounded ω-psh function such that $\sup_X \varphi = 0$ and

$$(\omega + dd^c\varphi)^n = e^{-\varepsilon\psi + c_\psi} f\omega^n.$$

The density on the right-hand side satisfies Condition $\mathcal{S}(B,\alpha)$ since $c_\psi \le 0$ and since $e^{\varepsilon C} \le 2$. We thus get from Lemma 5.3.4 below that $\varphi \ge -C$. Thus we have defined a mapping from $\mathcal{C}$ to itseft

$$\Phi : \mathcal{C} \to \mathcal{C}, \quad \Phi(\psi) := \varphi.$$

Let us prove that Φ is continuous on $\mathcal{C}$. Let ψ_j be a sequence in $\mathcal{C}$ which converges to ψ in $L^1(X)$. Denote by

$$c_j := c_{\psi_j}, \ \ c := c_\psi, \ \Phi(\psi_j) = \varphi_j, \ \Phi(\psi) = \varphi.$$

It is enough to prove that any cluster point of the sequence (φ_j) is equal to φ. Therefore, we can assume that φ_j converges to φ_0 in $L^1(X)$ and up to extracting a subsequence that ψ_j converges almost everywhere to ψ on X and also that c_j converges to $c_0 \in [-C\varepsilon, 0]$. Since $e^{-\varepsilon\psi_j + c_j} f$ converges in $L^1(X)$ to $e^{-\varepsilon\psi + c_0} f$ in $L^1(X)$ and almost everywhere, it follows from [DNL14a, Lemma 3.4] that

$$(\omega + dd^c\varphi_0)^n = e^{-\varepsilon\psi + c_0} f\omega^n.$$

It is clear that $c_0 = c$ and it follows from Hartogs' lemma that $\sup_X \varphi_0 = 0$. Thus $\varphi_0 = \varphi$. This concludes the continuity of Φ.

Now, since $\mathcal{C}$ is compact and convex in $L^1(X)$ and since Φ is continuous on $\mathcal{C}$, by Schauder Fixed Point Theorem there exists a fixed point of Φ, say φ. Then $\varphi - c_\varphi/\varepsilon$ is the desired solution. □

We refer the reader to [DNL14a, Section 4.2] for the proof of the following lemma.

Lemma 5.3.4. *Assume that f satisfies Condition $\mathcal{S}(B,\alpha)$ with $\alpha > 1$, $B > 0$. Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique function such that*

$$(\omega + dd^c\varphi)^n = f\omega^n, \ \sup_X \varphi = 0.$$

Then $\varphi \ge -C$ with $C = C(B,\alpha) > 0$.

5.3.3 Proof of Theorem C

Assume that $\varphi \in \mathcal{E}(X,\omega)$ satisfies

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\omega^n, \lambda > 0.$$

Up to rescaling ω it suffices to treat the case when $\lambda = 1$. The proof of Theorem C is quite similar to that of Theorem B. The difference here is that f is not integrable. For convenience of the reader we rewrite the arguments here. Since f satisfies Condition $\mathcal{H}_f$ we can write $\log f = \psi^+ - \psi^-$, where

$\psi^{\pm}$ are qpsh functions on X, ψ^- is locally bounded on $X \setminus D$ and there exists a uniform constant $C > 0$ such that

$$dd^c\psi^{\pm} \geq -C\omega, \ \sup_X \psi^+ \leq C.$$

We apply the smoothing kernel ρ_ε in Demailly regularization theorem [Dem92] to the functions φ and $\psi^{\pm}$. For ε small enough, we get

$$dd^c\rho_\varepsilon(\psi^-) \geq -C_1\omega, \ \ dd^c\rho_\varepsilon(\varphi+\psi^+) \geq -C_1\omega, \ \ \sup_X \rho_\varepsilon(\varphi+\psi^+) \leq C_1,$$

where C_1 depends on C, the Lelong numbers of the currents $C\omega + dd^c\psi^{\pm}$ and $\sup_X \varphi$. By the classical result of Yau [Yau78], for each ε, there exists a unique smooth ω-psh function ϕ_ε satisfying

$$\mathrm{MA}\,(\phi_\varepsilon) = e^{c_\varepsilon+\rho_\varepsilon(\varphi+\psi^+)-\rho_\varepsilon(\psi^-)}\omega^n = g_\varepsilon\omega^n, \ \ \sup_X \phi_\varepsilon = 0,$$

where c_ε is a normalization constant such that

$$\int_X g_\varepsilon\omega^n = \int_X e^{\varphi} f\omega^n = \int_X \omega^n.$$

Since by Jensen's inequality $e^{\rho_\varepsilon(\varphi+\log f)} \leq \rho_\varepsilon(e^{\varphi+\log f})$ and $e^{\rho_\varepsilon(\varphi+\log f)}$ converges point-wise to $e^{\varphi}f$ on X, it follows from the general Lebesgue dominated convergence theorem that $e^{\rho_\varepsilon(\varphi+\log f)}$ converges to $e^{\varphi}f$ in $L^1(X)$ when $\varepsilon \downarrow 0$. This means that c_ε converges to zero when $\varepsilon \to 0$. It then follows from Lemma 3.4 in [DNL14a] that ϕ_ε converges in $L^1(X)$ to $\varphi - \sup_X \varphi$. We now apply the $\mathcal{C}^2$ estimate in Theorem 3.2 in [DNL14a] to get

$$n + \Delta\phi_\varepsilon \leq C_3 e^{-2\rho_\varepsilon(\psi^-)} \leq C_4 e^{-2\psi^-},$$

where C_3, C_4 are uniform constants (do not depend on ε). Now, we need to bound φ from below. By the assumption on f we have

$$\mathrm{MA}\,(\varphi) = e^{\varphi+\psi^+-\psi^-}\omega^n \leq e^{-(\psi^- - C_1)}\omega^n.$$

Consider $\psi := \frac{1}{2C}\psi^-$. Since this function belongs to $\mathrm{PSH}(X, \omega/2)$ we can apply Lemma 5.3.1 to get

$$\varphi - \sup_X \varphi \geq \psi - C_5.$$

Now the remaining part of the proof follows by exactly the same way as we have done in [DNL14a, Section 3.3].

5.3.4 Non Integrable densities

When $0 \leq f \notin L^1(X)$ it is not clear that we can find a solution $\varphi \in \mathcal{E}(X,\omega)$ of equation

$$(\omega + dd^c\varphi)^n = e^{\varphi} f\omega^n.$$

We show in the following that it suffices to find a subsolution. Another similar result has been proved by Berman and Guenancia in [BG13] using the variational approach. We provide here a simple proof using our generalized Monge-Ampère capacities.

Theorem 5.3.5. *Let $0 \leq f$ be a measurable function such that $\int_X f\omega^n = +\infty$. If there exists $u \in \mathcal{E}(X,\omega)$ such that $\mathrm{MA}\,(u) \geq e^u f\omega^n$ then there is a unique $\varphi \in \mathcal{E}(X,\omega)$ such that*

$$\mathrm{MA}\,(\varphi) = e^{\varphi} f\omega^n.$$

Proof. The uniqueness follows easily from the comparison principle. Indeed, one can find a proof in [BG13, Proposition 3.1]. We now establish the existence. For each $j \in \mathbb{N}$ we can find $\varphi_j \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ such that

$$(\omega + dd^c\varphi_j)^n = e^{\varphi_j} \min(f,j)\omega^n.$$

It follows from the comparison principle that φ_j is non-increasing and $\varphi_j \geq u$. Then $\varphi_j \downarrow \varphi \in \mathcal{E}(X,\omega)$ and by continuity of the complex Monge-Ampère operator along decreasing sequence in $\mathcal{E}(X,\omega)$ we get

$$\mathrm{MA}\,(\varphi) = e^{\varphi} f\omega^n.$$

Indeed, since $\mathrm{MA}\,(\varphi_j)$ converges weakly to $\mathrm{MA}\,(\varphi)$, from Fatou's lemma we get

$$\mathrm{MA}\,(\varphi) \geq e^{\varphi} f\omega^n$$

in the sense of positive Borel measures. To get the reverse inequality we need to show that the right-hand side has full mass, i.e.

$$\int_X e^{\varphi} f\omega^n = \int_X \omega^n.$$

Fix $\varepsilon > 0$. Since φ is ω-psh, in particular quasi-continuous, we find U an open subset of X such that $\mathrm{Cap}_\omega(U) < \varepsilon$ and φ is continuous on $K := X \setminus U$. Then φ is bounded on K and hence f must be integrable on K. We thus can apply the Lebesgue Dominated Convergence Theorem on K to get

$$\lim_{j\to+\infty} \int_K \mathrm{MA}\,(\varphi_j) = \lim_{j\to+\infty} \int_K e^{\varphi_j} \min(f,j)\omega^n = \int_K e^{\varphi} f\omega^n.$$

We can assume that $\varphi_j \leq 0$. It follows from Theorem 5.1.9 that

$$\int_U \mathrm{MA}\,(\varphi_j) \leq \mathrm{Cap}_{u,0}(U) \leq F(\varepsilon) \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

This implies that

$$\begin{aligned}\int_X e^{\varphi} f\omega^n &\geq \int_K e^{\varphi} f\omega^n = \lim_{j\to+\infty}\int_K \mathrm{MA}\,(\varphi_j)\\ &= \int_X \mathrm{MA}\,(\varphi_j) - \lim_{j\to+\infty}\int_U \mathrm{MA}\,(\varphi_j)\\ &\geq \int_X \omega^n - F(\varepsilon).\end{aligned}$$

By letting $\varepsilon \to 0$ we get $\int_X e^{\varphi} f\omega^n = \int_X \omega^n$, which completes the proof. □

Remark 5.3.6. Theorem 5.3.5 also holds if ω is merely semipositive and big.

Example 5.3.7. Let $D = \sum_{j=1}^N D_j$ be a simple normal crossing divisor on X. Assume that the D_j are defined by $s_j = 0$, where s_j are holomorphic sections such that $|s_j| < 1/e$. Consider the following density

$$f = \frac{1}{\prod_{j=1}^N |s_j|^2}.$$

Then for suitable positive constants C_1, C_2 the following function

$$\varphi := -2\sum_{j=1}^N \log(-\log|s_j| + C_1) - C_2$$

is a subsolution of $\mathrm{MA}\,(\varphi) = e^{\varphi} f\omega^n$. In fact, it suffices to find a function $u \in \mathcal{E}(X, \omega/2)$ such that $e^u f$ is integrable (see Example 5.3.9).

5.3.5 The case of semipositive and big classes

In this section we try to extend our result in Theorem C to the case of semipositive and big classes. Let θ be a smooth closed semipostive $(1,1)$-form on X such that $\int_X \theta^n > 0$. Assume that $E = \sum_{j=1}^M a_j E_j$ is an effective simple normal crossing divisor on X such that $\{\theta\} - c_1(E)$ is ample. Let $0 \leq f$ is a non-negative measurable function on X. Consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = e^{\varphi} f\omega^n. \tag{5.3.3}$$

As in Theorem C we obtain here a similar regularity for solutions in $\mathcal{E}(X, \omega)$:

Theorem 5.3.8. *Assume that $0 < f \in \mathcal{C}^\infty(X \setminus D)$ satisfies Condition $\mathcal{H}_f$. Let θ and E be as above. If there is a solution in $\mathcal{E}(X, \omega)$ of equation (5.3.3) then this solution is also smooth on $X \setminus (D \cup E)$.*

Note that in Theorem 5.3.8 we do not assume that f is integrable on X. We also stress that there is at most one solution in $\mathcal{E}(X, \theta)$ (see [BG13]).

Proof. We adapt the proof of Theorem 3 in [DNL14a] where we followed essentially the ideas in [BEGZ10]. Assume that $\varphi \in \mathcal{E}(X,\theta)$ is a solution to equation (5.3.3). By assumption on f we can find a uniform constant $C > 0$ such that

$$f = e^{\psi^+ - \psi^-}, \ dd^c\psi^\pm \geq -C\omega^n, \ \sup_X \psi^+ \leq C, \ \sup_X \varphi \leq C, \ \psi^- \in L^\infty_{\rm loc}(X \setminus D).$$

We regularize φ and $\psi^\pm$ by using the smoothing kernel ρ_ε in Demailly's work [Dem92]. Then for $\varepsilon > 0$ small enough we have

$$dd^c\rho_\varepsilon(\psi^-) \geq -C_1\omega, \ dd^c\rho_\varepsilon(\varphi+\psi^+) \geq -C_1\omega, \ \sup_X \rho_\varepsilon(\varphi+\psi^+) \leq C_1,$$

where C_1 depends on C and the Lelong numbers of the currents $C\omega + dd^c\psi^\pm$. For each $\varepsilon > 0$ by the famous result of Yau [Yau78] there exits a unique smooth $\phi_\varepsilon \in \mathrm{PSH}(X, \theta+\varepsilon\omega)$ normalized by $\sup_X \phi_\varepsilon = 0$ such that

$$(\theta + \varepsilon\omega + dd^c\phi_\varepsilon)^n = e^{c_\varepsilon + \varphi_\varepsilon + \psi_\varepsilon^+ - \psi_\varepsilon^-}\omega^n = g_\varepsilon \omega^n,$$

where c_ε is a normalized constant. As in the proof of Theorem 3 in [DNL14a] we can prove that c_ε converges to 0 as $\varepsilon \downarrow 0$. We then can show that ϕ_ε converges in L^1 to $\varphi - \sup_X \varphi$. Now, we can apply Theorem 5.1 and Theorem 5.2 in [DNL14a] to get uniform bound on ϕ_ε and $\Delta_\omega\phi_\varepsilon$ on every compact subset of $X \setminus (D \cup E)$. From this we can get the smoothness of φ on $X \setminus (D \cup E)$ as in [DNL14a]. □

It follows from Theorem 5.3.5 (which is also valid in the case of semipositive and big classes) that to solve the equation it suffices to find a subsolution in $\mathcal{E}(X,\theta)$. We show in the following example that in some cases it is easy to find a subsolution in $\mathcal{E}(X,\theta)$.

Example 5.3.9. We consider the density given in Example 5.3.7. Assume that θ satisfies $\{\theta\} - c_1(E) > 0$, where $E = \sum_{j=1}^M a_jE_j$ is an effective simple normal crossing divisor on X. Assume that E_j is defined by the zero locus of a holomorphic section σ_j such that $|\sigma_j| < 1/e$. Then for some constants $p \in (0,1)$ and $a > 0$, $A \in \mathbb{R}$ the following function

$$u := -\left(-a\sum_{j=1}^N \log|s_j| - \frac{1}{2}\sum_{j=1}^M a_j \log|\sigma_j|\right)^p - A$$

belongs to $\mathcal{E}(X,\theta/2)$ and verifies $\int_X e^u f\omega^n = 2^{-n}\int_X \theta^n$. It follows from [BEGZ10] that there exists $v \in \mathcal{E}(X,\theta/2)$ such that $v \leq 0$ and

$$(\theta/2 + dd^cv)^n = e^u f\omega^n.$$

It is easy to see that $\varphi := u + v \in \mathcal{E}(X,\theta)$ is a subsolution of (5.3.3).

5.3.6 Critical Integrability

Recently, Berndtsson [Ber13] solved the openness conjecture of Demailly and Kollár [DK01] which says that given $\phi \in \mathrm{PSH}(X,\omega)$ and

$$\alpha(\phi) = \sup\{t > 0 \mid e^{-t\phi} \in L^1(X)\} < +\infty,$$

then one has $e^{-\alpha\phi} \notin L^1(X)$ (a stronger version of the openness conjecture has been quite recently obtained by Guan and Zhou [GZ13]).

In the following result, we use the generalized capacity to show that $e^{-\alpha\phi}$ is however not far to be integrable in the following sense:

Theorem 5.3.10. *Let $\phi \in \mathrm{PSH}(X,\omega)$ and $\alpha = \alpha(\phi) \in (0,+\infty)$ be the canonical threshold of ϕ. Then we can find $\varphi \in \mathrm{PSH}(X,\omega)$ having zero Lelong number at all points of X such that*

$$\int_X e^{\varphi - \alpha\phi}\omega^n < +\infty.$$

In what follows we give a proof of the above result that uses generalized Monge-Ampère capacities. However, using a constructive proof, one can chose $\varphi = \chi \circ \phi \in \mathcal{E}(X,\omega)$ for some χ increasing convex function.

Proof. Let α_j be an increasing sequence of positive numbers which converges to α. By assumption we have $e^{-\alpha_j\phi}$ is integrable on X. We can assume that $\phi \leq 0$. We solve the complex Monge-Ampère equation

$$(\omega + dd^c\varphi_j)^n = e^{\varphi_j - \alpha_j\phi}\omega^n.$$

For each j, since $e^{-\alpha_j\phi}$ belongs to L^{p_j} for some $p_j > 1$, it follows from the classical result of Kołodziej [Koł98] that φ_j is bounded. Moreover, the comparison principle reveals that φ_j is non-increasing. Now, we need to bound φ_j uniformly from below by some singular quasi-psh function.

Let $1/2 > \varepsilon > 0$ be a very small positive number. By assumption we know that

$$e^{(\varepsilon-\alpha)\phi} \in L^p(X), \quad p = p_\varepsilon := \frac{\alpha - \varepsilon/2}{\alpha - \varepsilon} > 1.$$

Set $\psi := \varepsilon\phi \in \mathrm{PSH}(X,\omega/2)$ and consider the function

$$H(t) := \left[\mathrm{Cap}_\psi(\varphi_j < \psi - t)\right]^{1/n}, \quad t > 0.$$

It follows from [DNL14a, Lemma 2.7] that $\mathrm{Cap}_\omega \leq 2^n \mathrm{Cap}_\psi$. By a strong volume-capacity domination in [GZ05, Remark 5.10] we also have

$$\mathrm{vol}_\omega \leq \exp\left(\frac{-C_1}{\mathrm{Cap}_\omega^{1/n}}\right),$$

where C_1 depends only on (X, ω). Fix $t > 0, s \in [0,1]$. Using [DNL14a, Proposition 2.8] and Hölder inequality we get

$$\begin{aligned} s^n \mathrm{Cap}_\psi(\{\varphi_j < \psi - t - s\}) &\leq \int_{\{\varphi_j < \psi - t\}} \mathrm{MA}\,(\varphi_j) \\ &\leq \int_{\{\varphi_j < \psi - t\}} e^{-\varphi_j} e^{\psi} \mathrm{MA}\,(\varphi_j) \\ &\leq \int_{\{\varphi_j < \psi - t\}} e^{(\varepsilon - \alpha)\phi} \omega^n \\ &\leq \left[\int_X e^{(\varepsilon/2 - \alpha)\phi} \omega^n\right]^{1/p} \left[\int_{\{\varphi_j < \psi - t\}} \omega^n\right]^{1/q} \\ &\leq C_2 \left[\mathrm{Cap}_\psi(\{\varphi_j < \psi - t\})\right]^2, \end{aligned}$$

where $p = p_\varepsilon := \frac{\alpha - \varepsilon/2}{\alpha - \varepsilon} > 1$ and $q > 1$ is the exponent conjugate of p. The constant $C_2 > 0$ depends on ε and also on $\int_X e^{(\varepsilon/2 - \alpha)\phi} \omega^n$. We then get

$$sH(t+s) \leq C_2^{1/n} H(t)^2, \ \forall t > 0, \forall s \in [0,1].$$

Then by [EGZ09, Lemma 2.4] we get

$$\varphi_j \geq \varepsilon \phi - C_\varepsilon,$$

where C_ε only depends on ε and $\int_X e^{(\varepsilon/2 - \alpha)\phi} \omega^n$. Then we see that φ_j decreases to $\varphi \in \mathrm{PSH}(X, \omega)$ and φ satisfies

$$\varphi \geq \varepsilon \phi - C_\varepsilon.$$

Since ε is arbitrarily small we conclude that φ has zero Lelong number everywhere on X. Finally, it follows from Fatou's lemma that $e^{\varphi - \alpha\phi}$ is integrable on X. □

We now show with an indipendent and constructive argument that φ can be chosen to be in $\mathcal{E}(X, \omega)$, more precisely $\varphi = \chi \circ \phi$,

$$\int_X e^{\chi \circ \phi - \alpha \phi} \omega^n < +\infty,$$

for some $\chi : \mathbb{R}^- \to \mathbb{R}^-$ increasing convex function such that $\chi(-\infty) = -\infty$ and $\chi'(-\infty) = 0$. Note that $\chi \circ \phi \in \mathcal{E}(X, \omega)$ thanks to [CGZ08]. We are grateful to H. Guenancia for the following constructive proof.
We can always assume that $\phi \leq -1$. For each $k \in \mathbb{N}$ let

$$a_k := \log \int_X e^{-(\alpha - 2^{-k-1})\phi} \omega^n < +\infty. \tag{5.3.4}$$

Define the sequence (c_k) inductively by

$$c_1 = a_1,\ c_{k+1} := \max(c_k + 4k, a_{k+1}),\ \forall k \geq 1. \tag{5.3.5}$$

Define another sequence (t_k) by

$$t_1 := 1,\ t_{k+1} := 2^{k+1}(c_{k+1} - c_k),\ \forall k \geq 1. \tag{5.3.6}$$

Define $\chi : (-\infty, -1] \to \mathbb{R}^-$ by

$$\chi(-t) := -2^{-k}t - c_k \quad \text{if} \quad t \in [t_k, t_{k+1}],\ \forall k \geq 1.$$

It follows from (5.3.4) that

$$e^{(\alpha - 2^{-k-1})t} \ \text{vol}(\phi < -t) \leq \int_X e^{-(\alpha - 2^{-k-1})\phi}\omega^n \leq e^{c_k}.$$

Thus using (5.3.5), (5.3.6) and the above inequality we get

$$\begin{aligned}
\int_X e^{\chi(\phi) - \alpha\phi}\omega^n &\leq e^{\chi(-1)+\alpha} + \alpha \int_1^{+\infty} e^{\alpha t + \chi(-t)} \,\text{vol}(\phi < -t)dt \\
&\leq C + \alpha \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} e^{\alpha t + \chi(-t)} \ \text{vol}(\phi < -t)dt \\
&\leq C + \alpha \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} e^{c_k + 2^{-k-1}t - 2^{-k}t - c_k} dt \\
&\leq C + \alpha \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} e^{-2^{-k-1}t} dt \\
&\leq C + \alpha \sum_{k=1}^{+\infty} 2^{k+1} e^{-2^{-k-1}t_k} \\
&\leq C + \alpha \sum_{k=1}^{+\infty} 2^{k+1} e^{-2^{-1}(c_k - c_{k-1})} \\
&\leq C + \alpha \sum_{k=1}^{+\infty} 2^{k+1} e^{-2(k-1)} \\
&\leq C + 4\alpha.
\end{aligned}$$

The above result is quite optimal as the following example shows:

Example 5.3.11. Let (X, ω) be a compact Kähler manifold and D be a smooth complex hypersurface on X defined by a holomorphic section s such that $|s| \leq 1/e$. Consider

$$\phi = 2\log|s| - (-\log|s|)^p, \quad p \in (0, 1). \tag{5.3.7}$$

By rescaling ω we can assume that $\phi \in \mathrm{PSH}(X,\omega)$. Then for any $q > 0$

$$\int_X \frac{e^{-\phi}}{(-\phi)^q}\omega^n = +\infty.$$

The example above has been given in [ACK+09] in the case of one complex variable which is locally similar to our setting. Assume now that ϕ is given by (5.3.7). It follows from Theorem 5.3.10 that we can find $\varphi \in \mathrm{PSH}(X,\omega)$ having zero Lelong number everywhere such that

$$\int_X e^{\varphi-\phi}\omega^n < +\infty.$$

In this concrete example one such function φ can be given explicitly by

$$\varphi = -(\log|s|)^p - (1+\varepsilon)\log(\log|s|),\ \varepsilon > 0.$$

Proof of Theorem D. It follows from the above proof of Theorem 5.3.10 that there exists $u \in \mathcal{E}(X,\omega/2)$ such that $e^{u-\alpha\phi}$ is integrable. We then can argue as in Example 5.3.9 to find a subsolution which also yields a solution thanks to Theorem 5.3.5. The uniqueness follows from the comparison principle (see [BG13]). □

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