Degenerate scale for 2D Laplace equation with Robin boundary condition

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ABSTRACT

It is well known that the 2D Laplace Dirichlet boundary value problem with a specific contour has a degenerate scale for which the boundary integral equation (BIE) has several solutions. We study here the case of the Robin condition (i.e. convection condition for thermal conduction problems), and show that this problem has also one degenerate scale. The cases of the interior problem and of the exterior problem are quite different. For the Robin interior problem, the degenerate scale is the same as for the Dirichlet problem. For the Robin exterior problem, the degenerate scale is always larger than for the Dirichlet problem and has some asymptotic properties. The cases of several simple boundaries like ellipse, equilateral triangle, square and rectangle are numerically investigated and the results are compared with the analytically predicted asymptotic behavior.

An important result is that avoiding a contour leading to a degenerate Robin problem cannot be achieved as simply as in the case of Dirichlet boundary condition by introducing a large reference scale into the Green's function.

1. Introduction

1.1. The case of Dirichlet and Neumann boundary conditions

The existence of a degenerate scale for the Laplace equation with Dirichlet in the plane is well known [1–9], for the Helmholtz problem [10] and also (with Dirichlet condition) for plane elasticity [11–16] and for the biharmonic equation [17–19]. For any C^2 simple closed curve Γ, there is a ρ scale such that there is a solution to the following exterior problem on $\rho\Gamma^-$, the exterior of $\rho\Gamma$ [20]:

$$\begin{cases} \Delta u = 0 \; x \in \rho\Gamma^-; \\ u(x) = 0 \; x \in \rho\Gamma. \end{cases} \tag{1}$$

with $u \in C^2(\rho\Gamma^-) \cap C^1(\rho\Gamma^- \cup \rho\Gamma)$ and with the radiative condition below that will be assumed in the following:

$$\begin{cases} u(x) = u(r,\theta) = -\dfrac{1}{2\pi}\ln(r) + \mathrm{O}(r^{-1}); \\ \dfrac{\partial u}{\partial r} = -\dfrac{1}{2\pi r} + \mathrm{O}(r^{-2}). \end{cases} \tag{2}$$

It can be seen that a non null solution u of (1) and (2) is also a solution of the BIE [20]:

$$\frac{1}{2}u(\mathbf{x}) + \int_{\rho\Gamma} G(\mathbf{x},\mathbf{y})q - u(\mathbf{y})\frac{\partial G}{\partial \mathbf{n}_y}\mathrm{d}S_y = 0 \; x \in \rho\Gamma,$$
$$\text{with } u = 0 \text{ on } \rho\Gamma;\; q = \frac{\partial u}{\partial \mathbf{n}};\; \int_{\rho\Gamma} q = 1;\; G(\mathbf{x},\mathbf{y}) = -1/2\pi\ln(|\mathbf{x}-\mathbf{y}|). \tag{3}$$

Reciprocally, a non null solution of (3) gives a non null solution of the exterior problem (1) and (2) thanks to the representation formula (e.g. [21]. Analogous results will be obtained in part 4 for the Robin problem.

The BIE for the interior Dirichlet problem is the same as for the exterior problem and so has non null solutions at the degenerate scale but the representation formula leads to a null solution for the interior problem. The discrepancy between the non-null solution of the exterior problem and the null solution of the interior problem has been examined using complex methods in [22]. The case of the interior problem with Robin condition will be investigated in part 3.

For Neumann boundary condition there is no such phenomenon. The interior BIE has a solution only if the compatibility condition $\int_\Gamma q\mathrm{d}S_y = 0$ is satisfied and the solution is unique up to an additive constant [23]. The exterior problem has also only one solution up to an additive constant. The radiative condition (2) can be fulfilled only if $\int_\Gamma q\mathrm{d}S_y = 1$.

1.2. The problem of the Robin boundary condition

We consider the following Robin condition for the interior problem:

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$$V_0\left(\frac{\partial u^0}{\partial \mathbf{n}} + tu^0\right) = 0. \tag{16}$$

If the boundary is not at its degenerate scale for Dirichlet condition, we conclude that $\frac{\partial u^0}{\partial \mathbf{n}} + tu^0 = 0$ and then u^0 is the solution of the interior problem with homogeneous Robin condition and so $u^0 = 0$ and u_0=0. Then Eq. (14) has only the null solution as proved in [23]. So, if the domain is at a degenerate scale for the interior Robin problem, it is at the degenerate scale for the Dirichlet problem.

3.2. If a domain is at the degenerate scale for the Dirichlet problem, it is also at a degenerate scale for the Robin problem

We consider now that the domain is at the degenerate scale for the Dirichlet problem. There is a function Φ such that $V_0(\Phi) = 0$ and $p(\Phi) = 1$ with $p(\Phi)$ the integral of Φ on Γ [11]. We consider now the problem:

$$\begin{cases} \Delta u = 0 \; x \in \Gamma^+; \\ \frac{\partial u}{\partial \mathbf{n}} + tu = \Phi \; x \in \Gamma. \end{cases} \tag{17}$$

It can be proved, that this problem (with $t > 0$) has a solution (for example by changing the scale and using integral methods (see [23])). Then this solution satisfies the standard BIE. Using the boundary condition in the BIE we get that the boundary value u_0 of u satisfies (14). We must also prove that u_0 is non null. Applying p to the boundary condition, we get: $p(\Phi) = 1 = p(\frac{\partial u_0}{\partial \mathbf{n}}) + tp(u_0) = tp(u_0)$ since $p(\frac{\partial u_0}{\partial \mathbf{n}}) = 0$, and so $u_0 \neq 0$.

Finding the degenerate scale for the interior Robin problem reduces to the well known problem of finding the degenerate scale for the Dirichlet problem as already shown for the case of a circle. From a mathematical view, this means evaluating the logarithmic capacity (e.g. [23]). Several numerical methods have also been investigated [22,29–31].

4. The general case of the exterior problem

This section is devoted to the mathematical study of the degenerate scale for the exterior problem. For a quick reading, the reader can go to Section 4.3 and note the conclusions of 4.4: there is one degenerate scale and this degenerate scale is a decreasing function of *t*.

4.1. Link between the BIE non null solution and a characterization of the degenerate scale for the exterior Robin problem

We consider possible non null solutions u_0 of (7). Using operator notations it writes out:

$$\left(-W_0 + \frac{1}{2}I + tV_0\right)(u_0) = 0. \tag{18}$$

Then we consider the following functions [23]:

$$u^- = tV^-(u_0) - W^-(u_0); \tag{19}$$

$$u^+ = tV^+(u_0) - W^+(u_0). \tag{20}$$

The functions u^- and u^+ are harmonic, u^- behaves at ∞ like $-\frac{t\int_\Gamma u_0}{2\pi}\ln|\mathbf{x}| + o(|\mathbf{x}|)$. Using the boundary values of V^+ and W^+ and the BIE (18) we find that $u^+ = 0$ on the boundary. Then, u^+ is the solution of the interior homogeneous Dirichlet problem and then: $u^+ = 0$. By applying $\frac{\partial}{\partial \mathbf{n}}$ and using the values of the normal derivative of V^+ and W^+ (see Appendix), we get:

$$\left(W_0^* + \frac{1}{2}\right)(tu_0) - N_0(u_0) = 0. \tag{21}$$

We consider now u^- and the boundary value of $\frac{\partial u^-}{\partial \mathbf{n}} - tu^-$. Using the properties of the normal derivatives of V^- and W^- (see appendix), we get:

$$\left(\frac{\partial u^-}{\partial \mathbf{n}} - tu^-\right) = \left(W_0^* - \frac{1}{2}I - N_0\right)(tu_0) - t\left(tV^-(u_0) - \left(W_0 + \frac{1}{2}\right)(u_0)\right). \tag{22}$$

It can be observed that $\frac{\int_\Gamma u_0}{2\pi} \neq 0$, elsewhere $u^-(x) \to 0$ if $|x| \to \infty$ and u^- is the unique solution of the exterior Robin problem tending to 0 at ∞ (see for example [23]) and is therefore null, and also u_0.

Then, $\frac{\int_\Gamma u_0}{2\pi} \neq 0$ and, up to a multiplicative factor, u^- is a solution of the following problem:

$$\begin{cases} \Delta u = 0 \; \mathbf{x} \in \Gamma^+; \\ \frac{\partial u}{\partial \mathbf{n}} - tu = 0 \; \mathbf{x} \in \Gamma; \end{cases} \tag{23}$$

with the radiative conditions (2).

Reciprocally, if u is a non null solution to the above problem, it satisfies the BIE (18).

4.2. Introduction of an auxiliary problem

We consider the exterior problem: to find u and a constant ω satisfying the following conditions:

$$\begin{cases} \Delta u = 0 \; \mathbf{x} \in \Gamma^-; \\ \frac{\partial u}{\partial \mathbf{n}} - tu = 0 \; \mathbf{x} \in \Gamma; \\ u = -\frac{1}{2\pi}\ln(r) + \frac{\omega}{2\pi} + O(r^{-1})\frac{\partial u}{\partial r} = -\frac{1}{2\pi r} + O(r^{-2}) \; r \to \infty. \end{cases} \tag{24}$$

Let us show now that the above problem (24) has a unique solution.

We refer to reference [23]; we assume that the curve Γ is C^2 and that $\mathbf{x} = 0 \notin \Gamma$. Then [23], the following problem has a unique solution (u, ω).

$$\begin{cases} u \in C^2(\Gamma^-) \cap C^1(\Gamma^- \cup \Gamma), \; \omega \in \mathbb{R}; \\ \Delta u = 0 \; \mathbf{x} \in \Gamma^-; \\ \frac{\partial u}{\partial \mathbf{n}} - tu = L(\mathbf{x}) \quad \mathbf{x} \in \Gamma, \; L \in C^1(\Gamma); \\ u = \frac{\omega}{2\pi} + O(r^{-1}); \; \frac{\partial u}{\partial r} = O(r^{-2})r \to \infty. \end{cases} \tag{25}$$

We solve the problem (25) with:

$$L(\mathbf{x}) = \frac{1}{2\pi}\left(\frac{\partial \ln|\mathbf{x}|}{\partial \mathbf{n}} - t\ln|\mathbf{x}|\right). \tag{26}$$

Then we see that $v = u - \frac{1}{2\pi}\ln|\mathbf{x}|$ is a solution to (24). This solution is unique, because the difference between two solutions of (24) is the solution of (25) with L=0 and $(u = 0, \omega = 0)$ is the unique solution to this problem.

4.3. Definition of the degenerate scale factor for an exterior Robin problem

Instead of thinking in terms of scaling of the problem, we think in terms of changing the unit length. This is equivalent for a Dirichlet problem, but not for a Robin problem because the term t is homogeneous to the inverse of a length. Dividing by ρ the unit of length, we define:

$$u_\rho(\mathbf{x}) = u(\rho\mathbf{x}); \; \Gamma_\rho^- = \rho\Gamma^-; \; \Gamma_\rho = \rho\Gamma. \tag{27}$$

The vector $\mathbf{n}$ is the outward pointing normal for the interior problem (Fig. 4). Then, if u is solution of (24), it can be seen that u_ρ is solution of

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Instead of thinking in terms of scaling of the problem, we think in terms of changing the unit length. This is equivalent for a Dirichlet problem, but not for a Robin problem because the term t is homogeneous to the inverse of a length. Dividing by ρ the unit of length, we define:

$$u_\rho(\mathbf{x}) = u(\rho\mathbf{x}); \; \Gamma_\rho^- = \rho\Gamma^-; \; \Gamma_\rho = \rho\Gamma. \tag{27}$$

The vector $\mathbf{n}$ is the outward pointing normal for the interior problem (Fig. 4). Then, if u is solution of (24), it can be seen that u_ρ is solution of

$$\{\Delta u_\rho = 0\ \mathbf{x} \in \Gamma_\rho^-;$$
$$\frac{\partial u_\rho}{\partial \mathbf{n}} - \frac{t}{\rho} u_\rho = 0\ \mathbf{x} \in \Gamma_\rho;$$
$$u_\rho = -\frac{1}{2\pi}\ln(r) - \frac{1}{2\pi}\ln\rho + \frac{\omega}{2\pi} + O(r^{-1});\ \frac{\partial u}{\partial r} = -\frac{1}{2\pi r} + O(r^{-2})\ r \to \infty. \tag{28}$$

The parameter t/ρ can be understood as the value of t in the new unit system. So we see that for $\rho = \exp\omega$, the problem has a solution with the asymptotic condition $u_\rho = \frac{-1}{2\pi}\ln(|\mathbf{x}|) + O(1/|\mathbf{x}|)$ when $|\mathbf{x}| \to \infty$. Then, the Eq. (23) has a non-null solution and Eq. (18) has also a non -null solution with the standard fundamental solution (see Section 4.1). This defines the degenerate scale. For $t \to \infty$, the problem tends to the Dirichlet problem. This is a generalization of the characterization proposed by Hsiao and Kleinmann [20] in the case of Dirichlet boundary condition. The existence and uniqueness are shown in [20] for Dirichlet problem and have been extended to the Robin case in Section 4.2.

4.4. The degenerate scale factor of the exterior problem is a decreasing function of t

We consider the problem of a circle of radius R_0 with Robin coefficient t_0. Substituting the parameters R_0, t_0 by ρR_0, t_0/ρ into (9), we get the following condition for the degenerate scale factor ρ:

$$\frac{t_0}{\rho}(\rho R_0)\ln(\rho R_0) = 1 \tag{29}$$

We get the following value of the degenerate scale factor:

$$\rho = \frac{1}{R_0} e^{1/(t_0 R_0)} \tag{30}$$

This example suggests that the degenerate scale factor is a decreasing function of t_0. In that section, we prove it.

Let us consider two values t_1, t_2 of t, with $t_1 > t_2 > 0$, and the corresponding solutions (u_1, ω_1), (u_2, ω_2). We consider also C_R, the circle of center O and radius R. We assume that R is large enough so that $\Gamma \subset C_R$. We denote by D_R the connected set bounded by C_R and Γ.

Using the Green's second formula we get:

$$-\int_{D_R} (u_1\Delta u_2 - u_2\Delta u_1)\mathrm{d}V = 0 = \int_{\partial D_R}\left(u_1\frac{\partial u_2}{\partial \mathbf{n}} - u_2\frac{\partial u_1}{\partial \mathbf{n}}\right)\mathrm{d}S. \tag{31}$$

(with $\mathbf{n}$ the interior normal of ∂D_R). The right hand term can be split into an integral I_1 on Γ and an integral I_2 on C_R. Using the Robin condition, we get:

$$I_1 = \int_\Gamma \left(u_1\frac{\partial u_2}{\partial \mathbf{n}} - u_2\frac{\partial u_1}{\partial \mathbf{n}}\right)\mathrm{d}S = \int_\Gamma \left(\frac{1}{t_2} - \frac{1}{t_1}\right)\frac{\partial u_1}{\partial \mathbf{n}}\frac{\partial u_2}{\partial \mathbf{n}}\mathrm{d}S. \tag{32}$$

Let us consider a point P where u_1 reaches its maximum value. This point is on the boundary of D_R because of the maximum principle applied to the harmonic function u_1. As u_1 decreases like $-\ln(R)/2\pi$ as $R \to \infty$, it is possible to find a sufficiently large value of R_0 so that the maximum is not reached on C_R if $R > R_0$ and therefore, it is reached on Γ.

If the maximum were strictly positive, using the boundary condition we would conclude that $\partial u_1/\partial \mathbf{n} > 0$ at this point of Γ and this point would not be a maximum. Then, u_1 is negative or null on Γ and therefore $\partial u_1/\partial \mathbf{n} \le 0$ on Γ.

The same applies to u_2. Taking into account the assumptions made about t_1 and t_2, we finally conclude: $I_1 \ge 0$.

Let us study the asymptotic behavior of I_2 when $R \to \infty$.

We have:

$$\begin{aligned} I_2 &= \int_{C_R}\left(u_1\frac{\partial u_2}{\partial \mathbf{n}} - u_2\frac{\partial u_1}{\partial \mathbf{n}}\right)\mathrm{d}S = \int_0^{2\pi}\left(u_1\frac{\partial u_2}{\partial \mathbf{n}} - u_2\frac{\partial u_1}{\partial \mathbf{n}}\right)R\mathrm{d}\theta \\ &= \int_0^{2\pi}\left[\left(-\frac{\ln R}{2\pi} + \frac{\omega_1}{2\pi} + O\left(\frac{1}{R}\right)\right)\left(\frac{1}{2\pi} + O\left(\frac{1}{R}\right)\right)\right] \\ &\quad - \left[\left(\frac{-\ln R}{2\pi} + \frac{\omega_2}{2\pi} + O\left(\frac{1}{R}\right)\right)\left(\frac{1}{2\pi} + O\left(\frac{1}{R}\right)\right)\right]\mathrm{d}\theta \\ &= \frac{(\omega_1 - \omega_2)}{2\pi} + O\left(\frac{\ln R}{R}\right). \end{aligned} \tag{33}$$

It is concluded that $I_2 = (\omega_1 - \omega_2)/2\pi$. As $I_1 = -I_2$ and $I_1 \ge 0$ if $t_1 > t_2 > 0$, we deduce that $\omega_2 \ge \omega_1$. We finally conclude that ω and ρ are decreasing functions of t. The Dirichlet condition corresponds to t infinite and it induces that the degenerate scale for the Dirichlet condition is smaller than for the Robin condition. In particular, if a contour is smaller than the one which is at the degenerate scale for Dirichlet condition, it is not at the degenerate scale for any Robin condition. The proof is also true if t_1 and t_2 are not constant on Γ.

5. Numerical search of degenerate scales for the Robin condition

5.1. Numerical methods

The numerical search for degenerate scales is based on the classical solution of boundary value problems by Boundary Element Methods (BEM). To build the searching method, the influence matrices $[G]$ and $[H]$ are produced for a given boundary Γ by using the solution of the Boundary Element Method with constant elements. These influence matrices relate the discretized displacements and normal fluxes at points along the boundary located at the centers of boundary elements by:

$$[H][u] - [G][q] = 0 \tag{34}$$

where $[u]$ is a column vector containing the values of the potential at nodes of the boundary and $[q]$ contains the normal derivatives of u at the same points. Conformly to usual practice in the field of BEM, the normal that is used in order to compute the normal derivatives is oriented outward the domain under consideration, i.e. $\mathbf{n}_e$ for the exterior problem and $\mathbf{n}$ for the interior problem. Therefore, if Γ is at a degenerate scale for the boundary condition, with the convention above for the normal, $[q] = -t[u]$ and $[u]$ is solution of the numerical system of linear equations:

$$([H] + t[G])[u] = 0. \tag{35}$$

5.1.1. Computation of the scaling factor

The solution of the boundary value problem (21) can be found by introducing the scaling factor ρ, leading to the expression of the scaled Green's function:

$$G = \frac{1}{2\pi}\ln\left(\frac{1}{\rho r}\right) = G_0 - \frac{1}{2\pi}\ln(\rho) = G_0 - C \tag{36}$$

with $G_0 = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$ and

$$\ln(\rho) = \omega = 2\pi C. \tag{37}$$

The numerical system becomes:

$$([H] + t[G] - tC[B])[u] = 0. \tag{38}$$

where $[B]$ is the matrix whose each line contains at column j the length of element j.

Finally, tC is the generalized eigenvalue of $([H] + t[G], [B])$. From a practical point of view, all eigenvalues but one are found very large and it is better suited to compute the generalized eigenvalues of $([B], [H] + t[G])$. All these eigenvalues are very small except one positive eigenvalue μ and $C = \frac{1}{t\mu}$. Finally, the scaling factor ρ is given by:

$$\rho = e^{\frac{2\pi}{t\mu}} \tag{39}$$

As shown previously, the contour $\rho\Gamma$ obtained by this procedure is at the degenerate scale for the convection constant equal to t/ρ. This method is quite similar to the computation of the smallest singular value of $[G]$ for the Dirichlet problem [3].

5.1.2. *Computation of the convection constant t_ρ making degenerate a given homothetical contour*

As seen before, starting from a given contour Γ which is degenerate for a Dirichlet boundary condition, every contour Γ_ρ being homothetical to Γ is degenerate for an unknown value t_ρ of the convection constant.

So, it is convenient to study all homothetical contours $\rho\Gamma$ and to find for any value of ρ the value of t for which $\rho\Gamma$ is degenerate. Starting from the numerical system of linear Eqs. (32) written for Γ, the system of linear equations for $\rho\Gamma$ can be obtained by replacing $[G]$ by $[G_\rho]$ given by:

$$[G_\rho] = \rho[G_0] - \frac{1}{2\pi}\rho\ln(\rho)[B], \tag{40}$$

Having built $[G_0]$, it is straightforward to build $[G_\rho]$ for any ρ.

Finally, the linear system becomes:

$$([H] + t\rho[G_0])[u] - t\frac{1}{2\pi}\rho\ln(\rho)[B] = 0 \tag{41}$$

and the value t_ρ of t for which the contour $\rho\Gamma$ is degenerate is the generalized eigenvalue of $([H], -\rho[G_0] + \frac{1}{2\pi}\rho\ln(\rho)[B])$.

It is important to notice that the factor ρ is no more considered as a change of unit length. Therefore the contour Γ_ρ is degenerate for t_ρ, which is not divided by ρ as for $\frac{t}{\rho}$ in the scaling procedure.

5.1.3. *Case of the interior problem*

Both methods work perfectly for the exterior problems. However, for the interior problem, it has been shown previously that the domain at the degenerate scale for Dirichlet boundary condition is at the degenerate scale for Robin condition and any value of t. The first method based on scaling gives this solution. Reversely, when looking for t_ρ related to a given value of ρ by using the second method, it has been found numerically that the generalized eigenvalue problem does not provide this result, which would correspond to a huge number of generalized eigenvalues. However, having computed the eigenvalues of $[H] + t[G_0]$ for different values of t, it has been verified that there is always a very small eigenvalue, of the order of 10^{-5} compared to other eigenvalues (for a meshing containing 500 boundary elements) and, therefore, it confirms that the domain which is degenerate for Dirichlet boundary condition is also degenerate for any t in the case of Robin boundary condition.

5.2. *Numerical examples*

For exterior domains, this method has been used for domains like: circles, squares, rectangles, ellipses, segments. In every case, the scaling factor ρ is computed for an initial domain which is degenerate for Dirichlet boundary condition. The discretization over the boundary was performed using 200 elements, giving a precision around 10^{-4}. For all these domains except rectangles, the analytical value of the degenerate scale for Dirichlet boundary condition is known, while for rectangles, the degenerate scales related to Dirichlet boundary condition for different shape factors have been obtained numerically, leading to the values of Fig. 2.

Knowing all degenerate scales for different boundaries, the previously described method has shown that all curves $\rho(t)$ have a trend similar to the one obtained for the circle, which is in $\rho = e^{1/t}$. Therefore, Fig. 3 shows the values of $\rho.\,e^{-1/t}$ as a function of t for a range of moderate values of t. This product is obviously equal to 1 fo the circle, is near to 1 for the ellipse and departs from this value for other

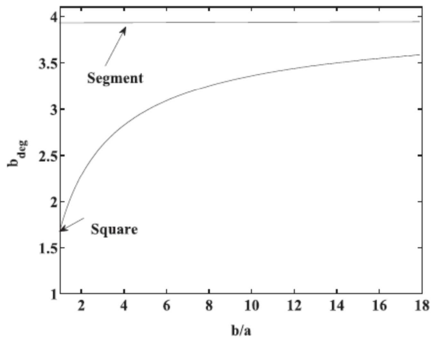

Fig. 2. The degenerate length b_{deg} of the long side of a rectangle for Dirichlet boundary condition as a function of the ratio between the long side and the short side.

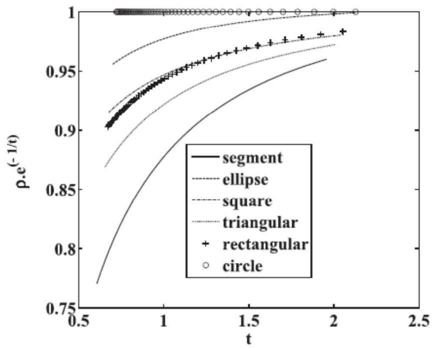

Fig. 3. The value of $\rho.\,e^{-1/t}$ as a function of t for various boundaries (medium values of t); the ratio of the sides of the rectangle and the ratio of the axes of the ellipse are both 3.

contours up to the case of the segment for which the value of $\rho.\,e^{-1/t}$ is 0.75 around t=0.6. For every case, the value of $\rho.\,e - 1/t$ is nearer to 1 for higher values of t.

The same method will allow us to obtain the numerical results for the asymptotic studies reported thereafter.

6. Introduction of a solution based on complex numbers for the exterior problem

6.1. *Robin boundary equation using a complex potential*

We use a method similar to the one used for obtaining exact degenerate scales in elasticity [13,15].

If u is a solution of an exterior Laplace problem in Γ^-, there exists a complex analytic, possibly multivalued, function f such that:

$$u(\mathbf{x}) = \frac{1}{2}(f(z) + \overline{f}(z)) \text{ with } z = x_1 + ix_2 \in \Gamma^- \tag{42}$$

We need to write the boundary condition for f (see Fig. 4).

$$\begin{aligned}\frac{\partial u}{\partial \mathbf{n}} &= \lim_{\epsilon\to 0}\frac{f(z+\epsilon e^{i\theta}) + \overline{f}(z+\epsilon e^{i\theta}) - f(z) - \overline{f}(z)}{2\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{f(z) + \epsilon e^{i\theta}f'(z) + \overline{f}(z) + \epsilon e^{-i\theta}\overline{f'}(z) - f(z) - \overline{f}(z) + 0(\epsilon)}{2\epsilon}\\ &= \frac{e^{i\theta}f'(z) + e^{-i\theta}\overline{f'}(z)}{2}.\end{aligned} \tag{43}$$

The Robin condition (23) writes out:

$$\Re(e^{i\theta}f'(z) - tf(z)) = 0. \tag{44}$$

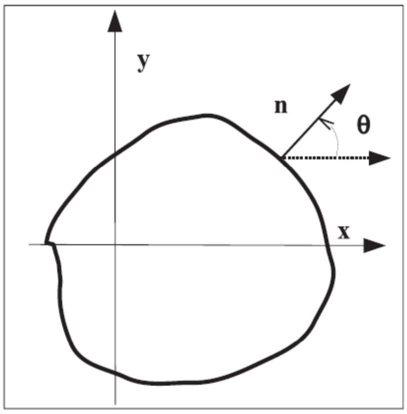

Fig. 4. Notations for the calculus of $\partial u/\partial \mathbf{n}$.

6.2. *Use of conformal mappings*

We consider a conformal mapping $w(z)$ from the outside of the unit circle C to the outside of the considered boundary Γ such that w has a Laurent series as follows:

$$w(z) = z + A_1/z + o(1/z). \tag{45}$$

Then Γ is at the degenerate scale for the Dirichlet condition (e.g. [22]). We consider now $W(z) = \rho w(z)$ and try to find ρ, the degenerate scale of Γ for the Robin condition (see Fig. 5). From the result in 4.4, we have always $\rho \geq 1$.

We assume that $\Phi(\zeta)$ is the solution for the exterior Robin problem with boundary Γ_ρ and with t/ρ; i.e. ρ is the degenerate scale for Γ and the Robin condition with the parameter t. The function $\Phi(\zeta)$ must satisfy the boundary Eq. (44) and the following condition at infinity

$$\Phi(\zeta) = -\frac{1}{2\pi}\ln(\zeta) - \Phi_0(\zeta), \tag{46}$$

$\Phi_0(\zeta)$ being a harmonic function tending to 0 at infinity.

Now, we have to find the value of $e^{i\theta}$ as a function of z. The tangent vector $\mathbf{t}$ is transformed as follows [32]: $\mathbf{t}_{\Gamma_R}(W(z)) = \mathbf{t}_C(z)\frac{W'(z)}{|W'(z)|}$. As the transform W is conformal and the orientation is unchanged, we have the same relation for the outwards normal vectors: $\mathbf{n}_{\rho\Gamma}(W(z)) = \mathbf{n}_C(z)\frac{w'(z)}{|w'(z)|}$. We deduce that:

$$e^{i\theta}(W(z)) = z\frac{w'(z)}{|w'(z)|}, \quad \text{for } |z| = 1. \tag{47}$$

Following the same approach as for elasticity [13,15], we define ϕ:

$$\phi(z) = \Phi(W(z)) = -\frac{1}{2\pi}\ln\rho - \frac{1}{2\pi}\ln(z) - \phi_0(z), \tag{48}$$

where $\phi_0(z)$ is a holomorphic function tending to 0 when $|z| \to \infty$. Substituting (47), (48) into (44), using $\Re(\ln(z)) = 0$ for $|z| = 1$, we get:

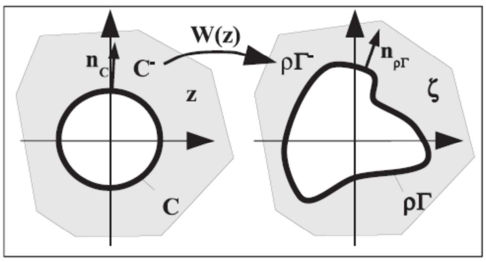

Fig. 5. Notations for conformal mapping.

$$\Re\left(-z\frac{1}{\rho|w'(z)|}\left(\frac{1}{2\pi z} + \phi'_0(z)\right) + \frac{1}{2\pi}\frac{t}{\rho}\ln\rho + \frac{t}{\rho}\phi_0(z)\right) = 0 \quad |z| = 1. \tag{49}$$

and finally, we have:

$$\Re\left(-z\frac{1}{|w'(z)|}\left(\frac{1}{2\pi z} + \phi'_0(z)\right) + \frac{1}{2\pi}t\ln\rho + t\phi_0(z)\right) = 0 \quad |z| = 1. \tag{50}$$

7. Asymptotic behavior for large t (small values of $\rho - 1$)

7.1. *Determination of the first parameters ρ_0, ρ_1 of the asymptotic expansion*

We write $\alpha = 1/t$ and we consider an asymptotic development $\rho = \rho_0 + \alpha\rho_1 + o(\alpha)$ with also $\phi_0 = \phi_{00} + \alpha\phi_{01} + o(\alpha)$ for $\alpha \to 0$. Substituting these expressions into (50), and multiplying by α we get:

$$\Re\left(\frac{\ln(\rho_0)}{2\pi} + \phi_{00}\right) + \alpha\Re\left(-\frac{z}{|w'|}\left(\frac{1}{2\pi z} + \phi'_{00}\right) + \frac{1}{2\pi}\frac{\rho_1}{\rho_0} + \phi_{01}\right) + \alpha^2(\ldots) = 0. \tag{51}$$

From (51), we get two equations:

$$\Re(\ln(\rho_0)/2\pi + \phi_{00}) = 0; \tag{52}$$

$$\Re\left(-\frac{z}{|w'|}\left(\frac{1}{2\pi z} + \phi'_{00}\right) + \frac{1}{2\pi}\frac{\rho_1}{\rho_0} + \phi_{01}\right) = 0. \tag{53}$$

We can write for a holomorphic function $g(z)$ on C^- continuous on C, such that $g(z)$ tends to $g(\infty)$ at ∞ (e.g. [33]):

$$\int_C g(z)\frac{dz}{iz} = 2\pi g(\infty). \tag{54}$$

Then we can also write:

$$\int_0^{2\pi}\Re(g(z))d\theta = \Re\left(\int_0^{2\pi} g(z)d\theta\right) = \Re\left(\int_C g(z)\frac{dz}{iz}\right) = \Re(2\pi g(\infty)). \tag{55}$$

Integrating (52), applying the relation (55) to ϕ_{00} and using that $\phi_{00}(\infty) = 0$, we deduce: $\ln\rho_0 = 0$; $\rho_0 = 1$, that is the classical result for the degenerate scale for the Dirichlet problem which is found by setting $t = \infty$. Then, we also deduce $\Re(\phi_{00}(z)) = 0$ for $z \in C$. Considering $\Re(\phi_{00}(1/z))$, we find that it is a harmonic function which is null on C^- and continuous. As a consequence, it is also null in the disk. Finally, we conclude $\phi_{00}(z) = ib$, $b \in \mathbb{R}$ and then $\phi'_{00}(z) = 0$. Then taking the integral of (53) on C, applying (55) to ϕ_{01} and using that $\phi_{01}(\infty) = 0$, we get:

$$\rho_1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|w'(e^{i\theta})|}d\theta. \tag{56}$$

For a circle we find $\rho_1 = 1$; then we deduce $\rho \simeq 1 + 1/t$ and that is consistent with (9) where we replace t by t/ρ and R by ρ.

7.2. *A lower bound for ρ_1*

Let us prove now that $\rho_1 \geq 1$.

We have:

$$\rho_1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|w'(e^{i\theta})|} \geq \frac{1}{2\pi}\left|\int_C \frac{dz}{zw'(z)}\right| \tag{57}$$

If $w'(z)$ is a holomorphic function which is non-null and continuous in $C^- \cup C$, then $1/w'$ is also holomorphic in C^- and continuous in $C^- \cup C$. Taking into account (45), we have $1/w'(\infty) = 1$. Then by (54), we have $\left|\int_C \frac{dz}{zw'(z)}\right| = 2\pi$ and we conclude $\rho_1 \geq 1$. This inequality is sharp since $\rho_1 = 1$ for the circle.

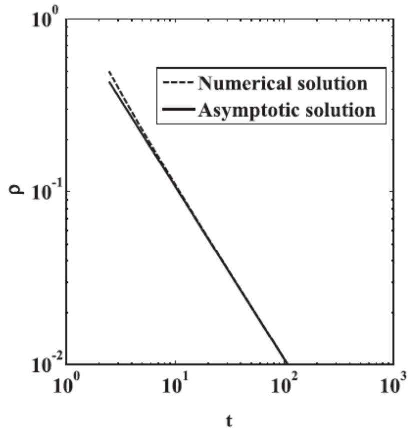

Fig. 6. Comparison between the asymptotic solution for large values of t and the numerical solution in the case of an ellipse with a ratio of radii=3 (log-log plot).

7.3. Case of ellipses

The ellipse is obtained by the conformal mapping $w(z) = z + a/z$. For a=1, the ellipse reduces to a segment. When $a \to 1$, the ellipse tends to a segment and $\rho_1 \to \infty$.

After integration of (56), the expression of ρ_1 is given by:

$$\rho_1 = \frac{2}{\pi(1+a)} K\left(\frac{2\sqrt{a}}{1+a}\right) \tag{58}$$

where K is the complete elliptic integral of first kind [34,35]. Fig. 6 shows for the case of an ellipse the values of the coefficient ρ and the value estimated by using the asymptotic formula for large values of t.

8. Asymptotic behavior for small t (large values of ρ)

8.1. Study of the parameter b governing the behavior of ρ for small t

Multiplying (50) by $|w'(z)|/\ln(\rho)$ we get:

$$\Re\left(-z\frac{1}{\ln(\rho)}\left(\frac{1}{2\pi z} + \phi'_0(z)\right) + \frac{t|w'(z)|}{2\pi} + t\frac{|w'(z)|\phi_0(z)}{\ln(\rho)}\right) = 0. \tag{59}$$

We assume that $\phi_0(z, t) = \phi_{00}(z) + o(t)$, $\phi'_0(z, t) = \phi'_{00}(z) + o(t)$ and look for an asymptotic expression for small values of t.

$$\frac{1}{\ln(\rho)} = bt + o(t), \tag{60}$$

with $b > 0$. We obtain:

$$\Re\left(-z(bt + o(t))\left(\frac{1}{2\pi z} + \phi'_{00}(z) + o(t)\right) + t\frac{|w'(z)|}{2\pi} + to(t)\right) = 0. \tag{61}$$

Considering the terms in t, we finally get the following condition:

$$\Re\left(-\frac{b}{2\pi} - bz\phi'_{00}(z) + \frac{|w'(z)|}{2\pi}\right) = 0. \tag{62}$$

As the function $\phi_{00}(z)$ tends to 0 when $|z| \to \infty$, it has an asymptotic expansion $\alpha/z + O(1/z^2)$ and the function ϕ'_{00} has an asymptotic expansion $-\alpha/z^2 + O(1/z^3)$ and then $z\phi'_{00} \to 0$ when $|z| \to \infty$. Assuming that ϕ'_{00} is continuous on $C^+ \cup C$ and applying (55), we get:

$$b = \frac{\int_0^{2\pi} |w'(e^{i\theta})| d\theta}{2\pi}. \tag{63}$$

In fact, it can be shown easily that the integral in (63) is the perimeter P of the curve Γ, so $b = P/2\pi$.

8.2. Lower bound for the coefficient b

We can give a lower bound of b assuming that w' is continuous on $C^+ \cup C$ and noting that $w'(z) \to 1$ as $|z| \to \infty$ [33]:

$$\int_0^{2\pi} |w'(e^{i\theta})| d\theta \geq \left|\int_0^{2\pi} w'(z)\frac{dz}{z}\right| = 2\pi. \tag{64}$$

We conclude $b \geq 1$ and the bound is sharp as it is attained for the circle. From the study of the n-armed star, it is readily seen that the coefficient b has no upper bound. Indeed, the perimeter of such stars can be made as large as chosen, if n is large enough.

8.3. Some exact values of the asymptotic coefficient b

For the circle of radius 1, we have:

$$b = 1, \tag{65}$$

and this is consistent with the exact formula (9).

We consider now an ellipse with semi axes $1 + a$, $1 - a$. Such an ellipse is at its degenerate scale since its logarithmic capacity is equal to $((1 + a) + (1 - a))/2 = 1$ [36]. The length of this ellipse is then given by the complete elliptic integral of the second kind [34,35]:

$$b = \frac{2(a+1)}{\pi} E\left(\frac{2\sqrt{a}}{1+a}\right). \tag{66}$$

Fig. 7 gives the graph of $b(a)$. It can be checked that for the circle a=0, $E(0) = \pi/2$ and then $b = 1$. For a segment, we have a=1, $E(1) = 1$ and then $b = 4/\pi$, as expected, because the perimeter of a segment is equal to twice its length.

For a regular n-polygon, we consider the Schwarz-Christoffel mapping $w(z) = \int_0^z (1 + \zeta^n)^{2/n} d\zeta$ [33,37] that maps the outside of the unit circle on the outside of the regular n-polygon. As the mapping $w(z)$ behaves like $z + O(1/z)$ when $|z| \to \infty$, the polygon is at its degenerate scale. We can find the length of one side of this n-polygon in [38] and we finally conclude:

$$b = \frac{2^{\frac{2}{n}}}{\pi} B\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right), \tag{67}$$

where B is the Beta function [35]. If n tends to ∞, we have $B(1/2, 1/2) = \pi$, and we recover the value for the circle b=1. (Fig. 7). Numerically, it can be found that $b \approx 1.1321$ for the equilateral triangle and $b \approx 1.0787$ for the square.

For a regular n-armed star with arms of length R the logarithmic capacity is: $R/4^{1/n}$ [36]. The condition for the degenerate scale for Dirichlet condition is $R = 4^{1/n}$. We deduce the value of b:

$$b = \frac{n4^{\frac{1}{n}}}{\pi}. \tag{68}$$

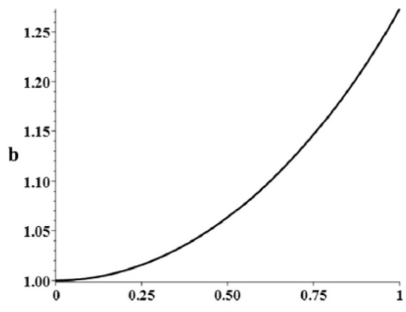

Fig. 7. Value of b as a function of the parameter a for an ellipse.

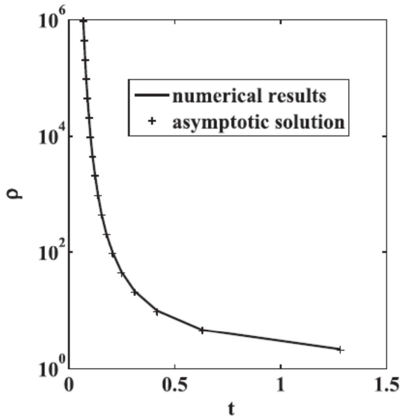

Fig. 8. Value of ρ as a function of t for an ellipse with ratio of radii equal to 3: comparison with the asymptotic solution for small values of t.

This value can be made as large as chosen, as noticed before.

Fig. 8 compares the asymptotic formula of ρ for large t with the numerically computed value for an ellipse whose ratio of radii is equal to 3.

8.4. The inclusion $\Gamma^+ \subset \Gamma'^+$ does not imply the inequality $\rho_\Gamma(t) \geq \rho_{\Gamma'}(t)$

The analogous implication holds for the Laplace Dirichlet problem [39] and there is a similar property for plane elasticity with Dirichlet condition [14]. But we can build a counter example in the Robin case. We consider the n armed star S_n with arm length equal to $4^{1/n}$. S_n is at its degenerate scale for the Dirichlet problem and we get from (60) and (68) the asymptotic value of the degenerate scale $\rho_{S_n}(t)$ for $t \to 0$:

$$\ln(\rho_{S_n})(t) \sim \frac{\pi}{n4^{1/n}}\frac{1}{t}. \tag{69}$$

We consider the circle C_n with radius $4^{1/n}$. The n armed star is included in C_n. Using (9) we find that the degenerate scale for C_n is

$$\ln(\rho_{C_n}) = \frac{1}{4^{1/n}}\frac{1}{t} - \ln(4^{1/n}) \tag{70}$$

Then:

$$\ln\left(\frac{\rho_{S_n}}{\rho_{C_n}}\right) \sim \left(\frac{\pi}{n4^{1/n}} - \frac{1}{4^{1/n}}\right)\frac{1}{t} \tag{71}$$

We conclude that for $n \geq 4$ and t small, we have $\rho_{S_n}(t) < \rho_{C_n}(t)$. It shows that the inclusion of a contour within another one does not lead always to the inverse inequality between the two degenerated scales. In this case, the curves $\rho(t)$ related to these two kinds of contours cross each other.

9. Conclusion

With Robin condition, there is one degenerate scale for interior problems and one degenerate scale for exterior problems, but these two cases are very different. For the interior problem, the degenerate scale is the same as for Dirichlet condition. For the exterior problem, the degenerate scale is always larger than the one related to Dirichlet condition. In that case, it has been possible to give the asymptotic behavior for small and large values of t. For small values of t the boundary condition tends to a Neumann condition and the degenerate scale tends to infinity (for Neumann condition there is no degenerate scale).

A numerical procedure has been described to obtain the degenerate scales by using influence matrices built from BEM formulation of the boundary value problem. Using this method, numerical computations have allowed to compare successfully the numerical values with the asymptotic formulas.

From a practical point of view, it can be emphasized that, contrarily to the case of Dirichlet boundary condition where the choice of a reference scale within the Green's function ensures the uniqueness of the solution, it is no more possible, because any chosen reference scale can lead to non-uniqueness in the case of Robin condition.

Appendix

We refer to [23,28] throughout this appendix.

Notations

We sum up here the definitions of the different integral operators. First, the single layer operators are defined by:

$$V^+(u) = \int_\Gamma u(x)G(x, y)\mathrm{d}S_y\ x \in \Gamma^+; \tag{72}$$

$$V^-(u) = \int_\Gamma u(x)G(x, y)\mathrm{d}S_y;\ x \in \Gamma^-; \tag{73}$$

$$V_0(u) = \int_\Gamma u(x)G(x, y)\mathrm{d}S_y;\ x \in \Gamma. \tag{74}$$

We also define the double layer operators, and the operator N_0:

$$W^+(u) = \int_\Gamma u(x)\frac{\partial G(x, y)}{\partial \mathbf{n}_y}\mathrm{d}S_y\ x \in \Gamma^+; \tag{75}$$

$$W^-(u) = \int_\Gamma u(x)\frac{\partial G(x, y)}{\partial \mathbf{n}_y}\mathrm{d}S_y\ x \in \Gamma^-; \tag{76}$$

$$W_0(u) = \int_\Gamma u(x)\frac{\partial G(x, y)}{\partial \mathbf{n}_y}\mathrm{d}S_y\ x \in \Gamma; \tag{77}$$

$$W_0^*(u) = \int_\Gamma u(x)\frac{\partial G(x, y)}{\partial \mathbf{n}_x} \mathrm{d}S_y \; x \in \Gamma. \tag{78}$$

The operator N_0 is defined as the normal derivative of W^+ on Γ.

Jump relations

$$V^+(u) = V^-(u) = V_0(u) \; x \in \Gamma; \tag{79}$$

$$W^+(u) = \left(W_0 - \frac{1}{2}\right)(u) \; x \in \Gamma; \tag{80}$$

$$W^-(u) = \left(W_0 + \frac{1}{2}\right)(u) \; x \in \Gamma. \tag{81}$$

Normal derivatives on the boundary

$$\partial_\mathbf{n} V^+(u) = \left(W_0^* + \frac{1}{2}\right)(u) \; x \in \Gamma; \tag{82}$$

$$\partial_\mathbf{n} V^-(u) = \left(W_0^* - \frac{1}{2}\right)(u) \; x \in \Gamma; \tag{83}$$

$$\partial_\mathbf{n} W^+(u) = \partial_\mathbf{n} W^-(u) = N_0(u) \; x \in \Gamma. \tag{84}$$

References

[1] Jaswon MA. Integral equation methods in potential theory, I. Proc R Soc Lond A 1963;235:23–32.
[2] Christiansen S. On two methods for elimination of non-unique solutions of an integral equation with a logarithmic kernel. Appl Anal 1982;13:1–18.
[3] Chen JT, Lin SR, Chen KH. Degenerate scale problem when solving Laplace's equation by BEM and its treatment. Int J Numer Methods Eng 2005;62(2):233–61.
[4] Corfdir A, Bonnet G. Degenerate scale for the Laplace problem in the half-plane; approximate logarithmic capacity for two distant boundaries. Eng Anal Bound Elem 2013;37:836–41.
[5] Vodička R. An asymptotic property of degenerate scales for multiple holes in plane elasticity. Appl Math Comput 2013;220:166–75.
[6] Vodička R, Petrík M. Degenerate scales for boundary value problems in anisotropic elasticity. Int J Solids Struct 2015;52:209–19.
[7] Chen YZ. Numerical solution for the degenerate scale in 2D Laplace equation for notch in half-plane using null field BIE. Eng Anal Bound Elem 2016;70:126–33.
[8] Chen JT, Lee YT, Chang YL, Jian J. A self-regularized approach for rank-deficiency systems in the BEM of 2D Laplace problems. Inverse Probl Sci Eng 2017;25(1):89–113.
[9] Li Z-C, Zhang L-P, Lee M-G. Interior field methods for Neumann problems of Laplace's equation in elliptic domains, comparisons with degenerate scales. Eng Anal Bound Elem 2016;71:190–202.
[10] Chen Jeng-Tzong, Lee Jia-Wei, Chen I -Lin, Po-Shen Kuo. On the null and nonzero fields for true and spurious eigenvalues of annular and confocal elliptical membranes. Eng Anal Bound Elem 2013;37(1):42–59.
[11] Constanda C. On non-unique solutions of weakly singular integral equations in plane elasticity. Q J Mech Appl Math 1994;47(2):261–8.
[12] Vodička R, Mantič V. On invertibility of elastic single-layer potential operator. J Elast 2004;74(2):147–73.
[13] Chen YZ, Lin XY, Wang ZX. Evaluation of the degenerate scale for BIE in plane elasticity by using conformal mapping. Eng Anal Bound Elem 2009;33:147–58.
[14] Corfdir A, Bonnet G. Variational formulation and upper bounds for degenerate scales in plane elasticity. J Elast 2015;118(2):209–21.
[15] Corfdir A, Bonnet G. Exact degenerate scales in plane elasticity using complex variable methods. Int J Solids Struct 2016;80:430–44.
[16] Chen JT, Huang WS, Lee YT, Kuo SR, Kao SK. Revisit of degenerate scales in the BIEM/BEM for 2D elasticity problems. Mech Adv Mater Struct 2017;24(1):1–15.
[17] Costabel M, Dauge M. Invertibility of the biharmonic single layer potential operator. Integral Equ Oper Theory 1996;24(1):46–67.
[18] Christiansen S. Derivation and analytical investigation of three direct boundary integral equations for the fundamental biharmonic problem. J Comput Appl Math 1998;91(2):231–47.
[19] Chen JT, Kuo SR, Chang YL, Kao SK. Degenerate-scale problem of the boundary integral equation method/boundary element method for the plate analysis. Eng Comput 2017, [accepted].
[20] Hsiao GC, Kleinmann RE. On a uniform characterization of capacity. In: Král J, editor. Potential theory. Prague, Czechoslovakia: Plenum Press; 1987. p. 103–26.
[21] Bonnet M. Boundary integral equations methods for solids and fluids. N.Y: Wiley; 1999.
[22] Kuo S-R, Chen J-T, Kao S-K. Linkage between the unit logarithmic capacity in the theory of complex variables and the degenerate scale in the BEM/BIEMs. Appl Math Lett 2013;26(9):929–38.
[23] Constanda C. Direct and indirect boundary integral equation methods. In: Monographs and surveys in pure and applied mathematics N107. Boca Raton: Chapman and Hall/CRC; 1999.
[24] Gilbarg D, Trudinger NS. Elliptic partial differential equations of second order. Berlin: Springer; 1998.
[25] Hahn DW, Özişik MN. Heat conduction, 3rd Edition. Hoboken, New Jersey: Wiley; 2012.
[26] Corless RM, Gonnet GH, Hare DEG, Jeffrey DJ, Knuth DE. On the Lambert W function. Adv Comput Math 1996;5:329–59.
[27] Dijkstra W, Mattheij RMM. A relation between the logarithmic capacity and the condition number of the BEM matrices. Commun Numer Methods Eng 2007;23:665–80.
[28] Hsiao GC, Wendland WL. Boundary integral equations. In: Applied mathematical sciences, vol. 164. Berlin: Springer; 2008.
[29] Chen JT, Kuo SR, Lin JH. Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity. Int J Numer Methods Eng 2002;54(12):1669–81.
[30] Chen J-T, Kuo S-R, Kao S-K, Jian J. Revisit of a degenerate scale: a semi-circular disc. J Comput Appl Math 2015;283:182–200.
[31] Chen YZ. Evaluation of the degenerate scale in antiplane elasticity using null field BIE. Appl Math Lett 2016;54:15–21.
[32] Bell SR. The cauchy transform, potential theory, and conformal mapping. Boca Raton: CRC Press; 1992.
[33] Muskhelishvili NI. Some basic problems of the mathematical theory of elasticity. Gronningen: Noordhoff; 1953.
[34] Boas ML. Mathematical methods in the physical sciences, 2nd Edition. N.Y.: Wiley; 1983.
[35] Abramowitz M, Stegun I. Handbook of mathematical functions with formulas, graphs, and mathematical tables. N.Y.: Dover; 1972.
[36] Rumely RS. Capacity theory on algebraic curves. Berlin: Springer-Verlag; 1989.
[37] Driscoll TA, Trefethen LN. Schwarz-Christoffel mapping, Cambridge monographs on applied and computational mathematics. Cambridge: Cambridge University Press; 2002.
[38] Kuo SR, Chen JT, Lee JW, Chen YW. Analytical derivation and numerical experiments of degenerate scales for regular N-gon domains in two-dimensional laplace problems. Appl Math Comput 2013;219:5668–83.
[39] Tsuji M. Potential theory in modern function theory. Tokyo: Maruzen; 1959.