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Abstract

A twisted generalization of Lie-Yamaguti algebras, called Hom-Lie-Yamaguti algebras, is defined. Hom-Lie-Yamaguti algebras generalize multiplicative Hom-Lie triple systems (and subsequently ternary multiplicative Hom-Nambu algebras) and Hom-Lie algebras in the same way as Lie-Yamaguti algebras generalize Lie triple systems and Lie algebras. It is shown that the category of (multiplicative) Hom-Lie-Yamaguti algebras is closed under twisting by self-morphisms. Constructions of Hom-Lie-Yamaguti algebras from ordinary Lie-Yamaguti algebras and Malcev algebras are given. Using the well-known classification of real two-dimensional Lie-Yamaguti algebras, examples of real two-dimensional Hom-Lie-Yamaguti algebras are given.

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1 Introduction

Using the Bianchi identities, K. Nomizu [15] characterized, by some identities involving the torsion and the curvature, reductive homogeneous spaces with some canonical connection. K. Yamaguti [19] gave an algebraic interpretation of these identities by considering the torsion and curvature tensors of Nomizu's canonical connection as a bilinear and a trilinear algebraic operations satisfying some axioms, and thus defined what he called a "general Lie triple system". M. Kikkawa [8] used the term "Lie triple algebra" for such an algebraic object. More recently, M.K. Kinyon and A. Weinstein [9] introduced the term "Lie-Yamaguti algebra" for this object.

A *Lie-Yamaguti algebra* $(V, *, \{,,\})$ is a vector space V together with a binary operation $* : V \times V \to V$ and a ternary operation $\{,,\} : V \times V \times V \to V$ such that

(LY1) $x * y = -y * x$,
(LY2) $\{x, y, z\} = -\{y, x, z\}$,
(LY3) $\circlearrowleft_{(x,y,z)}[(x * y) * z + \{x, y, z\}] = 0$,
(LY4) $\circlearrowleft_{(x,y,z)}\{x * y, z, u\} = 0$,
(LY5) $\{x, y, u * v\} = \{x, y, u\} * v + u * \{x, y, v\}$,
(LY6) $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + \{u, \{x, y, v\}, w\} + \{u, v, \{x, y, w\}\}$,

for all u, v, w, x, y, z in V and $\circlearrowleft_{(x,y,z)}$ denotes the sum over cyclic permutation of x, y, z.

In [2] the notation "LY-algebra" is used for "Lie-Yamaguti algebra". So, likewise, below we will write "Hom-LY algebra" for "Hom-Lie-Yamaguti algebra".

Observe that if $x * y = 0$, for all x, y in V, then $(V, *, \{,,\})$ reduces to a *Lie triple system* $(V, \{,,\})$ as defined in [18]. From the other hand, if $\{x, yz,\} = 0$ for all x, y, z in V, then $(V, *, \{,,\})$ is a Lie algebra $(V, *)$. Originally, N. Jacobson [7] defined a Lie triple system as a submodule of an associative algebra that is closed under the iterated commutator bracket.

In this paper we consider a Hom-type generalization of LY algebras that we call Hom-LY algebras. Roughly, a Hom-type generalization of a given type of algebras is defined by twisting the defining identities of that type of algebras by a self-map in such a way that, when the twisting map is the identity map, one recovers the original type of algebras. The systematic study of Hom-algebras was initiated by A. Makhlouf and S.D. Silvestrov [13], while D. Yau [23] gave a general construction method of Hom-type algebras starting from usual algebras and a twisting self-map. For information on various types of Hom-algebras, one may refer to [1], [6], [11]-[13], [22]-[25].

A Hom-type generalization of n-ary Lie algebras, n-ary Nambu algebras and n-ary Nambu-Lie algebras (i.e. Filippov n-ary algebras) called n-ary Hom-Lie algebras, n-ary Hom-Nambu algebras and n-ary Hom-Nambu-Lie algebras respectively, is considered in [1]. Such a generalization is extended to the one of Hom-Lie triple systems and Hom-Jordan triple systems in [25]. We point out that the class of (multiplicative) Hom-LY algebras encompasses the ones of multiplicative ternary Hom-Nambu algebras, multiplicative Hom-Lie triple systems (hence Jordan and Lie triple systems), Hom-Lie algebras (hence Lie algebras) and LY algebras.

The rest of the paper is organized as follows. In section 2 some basic facts on Hom-algebras and n-ary Hom-algebras are recalled. The emphasis point here is that the definition of a Hom-triple system (Definition 2.3) is more restrictive than the D. Yau's in [25]. However, with this vision of a Hom-triple system, we point out that any non-Hom-associative algebra (i.e. nonassociative Hom-algebra or Hom-nonassociative algebra) has a natural structure of (multiplicative) Hom-triple system (this is the Hom-counterpart of a similar well-known result connecting nonassociative algebras and triple systems). Then we give the definition of a Hom-LY algebra and make some observations on its relationships with some types of ternary Hom-algebras and with LY algebras. In section 3 we show that the category of Hom-LY algebras is closed under twisting by self-morphisms (Theorem 3.1). Subsequently, we show a way to construct Hom-LY algebras from LY algebras (or Malcev algebras) by twisting along self-morphisms (Corollary 3.2 and Corollary 3.3); this is an extension to binary-ternary algebras of a result due to D. Yau ([23], Theorem 2.3. Such an extension is first mentioned in [6], Corollary 4.5). In section 4 examples of real two-dimensional Hom-LY algebras are constructed, relaying on the classification of (real) two-dimensional LY algebras [5], [21].

All vector spaces and algebras throughout are considered over a ground field $\mathbb{K}$ of characteristic 0.

2 Ternary Hom-algebras. Definitions

We recall some basic facts about Hom-algebras, including ternary Hom-Nambu algebras. We note that the definition of a Hom-triple system given here (see Definition 2.3) is slightly more restrictive than the one given by D. Yau [25].

Then we give the definition of the main object of this paper (see Definition 2.6) and show its relationships with known structures such as ternary Hom-Nambu algebras, Hom-Lie triple systems, Hom-Lie algebras or Lie-Yamaguti algebras.

For definitions of n-ary Hom-algebras (n-ary Hom-Nambu and Hom-Nambu-Lie algebras, n-ary Hom-Lie algebras, etc.) we refer to [1], [25]. For information on origins of Nambu algebras, one may refer to [14], [17]. Here, for our purpose, we restrict our concern to ternary Hom-algebras. In fact, as we shall see below, a Hom-Lie-Yamaguti algebra is some multiplicative ternary Hom-Nambu algebra with an additional binary anticommutative operation satisfying some compatibility conditions.

Definition 2.1. ([25]). *A* **ternary Hom-algebra** $(V, [,,], \alpha = (\alpha_1, \alpha_2))$ *consists of a* $\mathbb{K}$*-module* V*, a trilinear map* $[,,] : V \times V \times V \to V$*, and linear maps* $\alpha_i : V \to V$*,* $i = 1, 2$*, called the* **twisting maps***. The algebra* $(V, [,,], \alpha = (\alpha_1, \alpha_2))$ *is said* **multiplicative** *if* $\alpha_1 = \alpha_2 := \alpha$ *and* $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$ *for all* $x, y, z \in V$*.*

For convenience, we assume throughout this paper that all Hom-algebras are multiplicative.

Definition 2.2. ([1]). *A* **(multiplicative) ternary Hom-Nambu algebra** *is a (multiplicative) ternary Hom-algebra* $(V, [,,], \alpha)$ *satisfying*

$$[\alpha(x), \alpha(y), [u, v, w]] = [[x, y, u], \alpha(v), \alpha(w)] + [\alpha(u), [x, y, v], \alpha(w)] + [\alpha(u), \alpha(v), [x, y, w]], \tag{2.1}$$

for all $u, v, w, x, y \in V$*.*

The condition (2.1) is called the *ternary Hom-Nambu identity*. In general, the ternary Hom-Nambu identity reads:

$$[\alpha_1(x), \alpha_2(y), [u, v, w]] = [[x, y, u], \alpha_1(v), \alpha_2(w)] + [\alpha_1(u), [x, y, v], \alpha_2(w)] + [\alpha_1(u), \alpha_2(v), [x, y, w]],$$

for all $u, v, w, x, y \in V$, where α_1 and α_2 are linear self-maps of V.

Definition 2.3. *A* **(multiplicative) Hom-triple system** *is a (multiplicative) ternary Hom-algebra* $(V, [,,], \alpha)$ *such that*

(i) $[u, v, w] = -[v, u, w]$*,*

(ii) $\circlearrowleft_{(u,v,w)}[u,v,w] = 0$,
for all $u, v, w \in V$.

Remark. *A more general definition of a Hom-triple system is given by D. Yau [25] without the requirements (i), (ii) as in Definition 2.3 above. Our definition here is motivated by the concern of giving a Hom-type analogue of the relationships between nonassociative algebras and triple systems (see Remark below after Proposition 2.4).*

A Hom-algebra in which the Hom-associativity is not assumed is called a nonassociative Hom-algebra [12] or a Hom-nonassociative algebra [22] (the expression of "non-Hom-associative" Hom-algebra is used in [6] for that type of Hom-algebras). With the notion of a Hom-triple system as above, we have the following

Proposition 2.4. *Any non-Hom-associative Hom-algebra is a Hom-triple system.*

Proof. Let $(A, \cdot, \alpha)$ be a non-Hom-associative algebra. Then $(A, [,], as(,,), \alpha)$ is a Hom-Akivis algebra with respect to $[x,y] := x \cdot y - y \cdot x$ (commutator) and $as(x,y,z) := xy \cdot \alpha(z) - \alpha(x) \cdot yz$ (Hom-associator), i.e. the Hom-Akivis identity

$$\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)] = \circlearrowleft_{(x,y,z)}as(x,y,z) - \circlearrowleft_{(x,y,z)}as(y,x,z)$$

holds for all x, y, z in A ([6]). Now define

$$[x,y,z] := [[x,y],\alpha(z)] - as(x,y,z) + as(y,x,z)$$

for all x, y, z in A. Then $[x,y,z] = -[y,x,z]$ and the Hom-Akivis identity implies that $\circlearrowleft_{(x,y,z)}[x,y,z] = 0$. Thus $(A, [,,], \alpha)$ is a Hom-triple system. □

Remark. *For* $\alpha = Id$ *(the identity map), we recover the triple system with ternary operation* $[[x,y],z] - (x,y,z) + (y,x,z)$ *that is associated to each nonassociative algebra, since any nonassociative algebra has a natural Akivis algebra structure with respect to the commutator and associator operations* $[x,y]$ *and* (x,y,z), *for all* x, y, z *(see, e.g., remarks and references in [6]).*

Definition 2.5. ([25]). *A* **Hom-Lie triple system** *is a Hom-triple system* $(V, [,,], \alpha)$ *satisfying the ternary Hom-Nambu identity (2.1).*

When $\alpha = Id$, a Hom-Lie triple system reduces to a Lie triple system.

We now give the definition of the basic object of this paper.

Definition 2.6. *A* **Hom-Lie-Yamaguti algebra** *(Hom-LY algebra for short) is a quadruple* $(L, *, \{,,\}, \alpha)$ *in which* L *is a* $\mathbb{K}$*-vector space, "*$*$*" a binary operation and "*$\{,,\}$*" a ternary operation on* L*, and* $\alpha : L \to L$ *a linear map such that*

(HLY1) $\alpha(x * y) = \alpha(x) * \alpha(y)$,

(HLY2) $\alpha(\{x, y, z\}) = \{\alpha(x), \alpha(y), \alpha(z)\}$,

(HLY3) $x * y = -y * x$,

(HLY4) $\{x, y, z\} = -\{y, x, z\}$,

(HLY5) $\circlearrowleft_{(x,y,z)}[(x * y) * \alpha(z) + \{x, y, z\}] = 0$,

(HLY6) $\circlearrowleft_{(x,y,z)}\{x * y, \alpha(z), \alpha(u)\} = 0$,

(HLY7) $\{\alpha(x), \alpha(y), u * v\} = \{x, y, u\} * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}$,

(HLY8) $\{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} = \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\} + \{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\} + \{\alpha^2(u), \alpha^2(v), \{x, y, w\}\}$,

for all u, v, w, x, y, z in L.

Note that the conditions (HLY1) and (HLY2) mean the multiplicativity of $(L, *, \{,,\}, \alpha)$.

Remark. *(1) If* $\alpha = Id$*, then the Hom-LY algebra* $(L, *, \{,,\}, \alpha)$ *reduces to a LY algebra* $(L, *, \{,,\})$ *(see (LY1)-(LY6)).*

(2) If $x * y = 0$*, for all* $x, y \in L$*, then* $(L, *, \{,,\}, \alpha)$ *becomes a Hom-Lie triple system* $(L, \{,,\}, \alpha^2)$ *and, subsequently, a ternary Hom-Nambu algebra (since, by Definition 2.5, any Hom-Lie triple system is automatically a ternary Hom-Nambu algebra).*

(3) If $\{x, y, z\} = 0$ *for all* $x, y, z \in L$*, then the Hom-LY algebra* $(L, *, \{,,\}, \alpha)$ *becomes a Hom-Lie algebra* $(L, *, \alpha)$*.*

3 Constructions of Hom-Lie-Yamaguti algebras

In this section we consider construction methods for Hom-LY algebras. These methods allow to find examples of Hom-LY algebras starting from ordinary LY algebras or even from Malcev algebras.

First, as the main tool, we show that the category of (multiplicative) Hom-LY algebras is closed under self-morphisms.

Theorem 3.1. *Let $A_\alpha := (A, *, \{,,\}, \alpha)$ be a Hom-LY algebra and let β be an endomorphism of the algebra $(A, *, \{,,\})$ such that $\beta\alpha = \alpha\beta$. Let $\beta^0 = id$ and, for any $n \geq 1$, $\beta^n := \beta \circ \beta^{n-1}$. Define on A the operations*
$x *_\beta y := \beta^n(x * y)$,
$\{x, y, z\}_\beta := \beta^{2n}(\{x, y, z\})$
for all x, y, z in A. Then $A_{\beta^n} := (A, *_\beta, \{,,\}_\beta, \beta^n\alpha)$ is a Hom-LY algebra, with $n \geq 1$.

Proof. First, we observe that the condition $\beta\alpha = \alpha\beta$ implies $\beta^n\alpha = \alpha\beta^n$, $n \geq 1$. Next we have
$(\beta^n\alpha)(x *_\beta y) = (\beta^n\alpha)(\beta^n(x) * \beta^n(y)) = \beta^n((\alpha\beta^n)(x) * (\alpha\beta^n)(y))$
$= (\alpha\beta^n)(x) *_\beta (\alpha\beta^n)(y) = (\beta^n\alpha)(x) *_\beta (\beta^n\alpha)(y)$ and we get (HLY1) for A_{β^n}. Likewise, the condition $\beta\alpha = \alpha\beta$ implies (HLY2). The identities (HLY3) and (HLY4) for A_{β^n} follow from the skew-symmetry of "$*$" and "$\{,,\}$" respectively.
Consider now $\circlearrowleft_{(x,y,z)}((x *_\beta y) *_\beta (\beta^n\alpha)(z)) + \circlearrowleft_{(x,y,z)}\{x, y, z\}_\beta$. Then

$$\begin{aligned}
\circlearrowleft_{(x,y,z)}((x *_\beta y) & *_\beta (\beta^n\alpha)(z)) + \circlearrowleft_{(x,y,z)}\{x, y, z\}_\beta \\
&= \circlearrowleft_{(x,y,z)}[\beta^n(\beta^n(x * y) * \beta^n(\alpha(z)))] \\
&+ \circlearrowleft_{(x,y,z)}[\beta^{2n}(\{x, y, z\})] \\
&= \circlearrowleft_{(x,y,z)}[\beta^{2n}((x * y) * \alpha(z))] + \circlearrowleft_{(x,y,z)}[\beta^{2n}(\{x, y, z\})] \\
&= \beta^{2n}(\circlearrowleft_{(x,y,z)}[(x * y) * \alpha(z) + \{x, y, z\}]) \\
&= \beta(0) \text{ (by (HLY5) for } A_\alpha) \\
&= 0
\end{aligned}$$

and thus we get (HLY5) for A_{β^n}. Next,
$\{x *_\beta y, (\beta^n\alpha)(z), (\beta^n\alpha)(u)\}_\beta = \{\beta^{3n}(x * y), \beta^{3n}(\alpha(z)), \beta^{3n}(\alpha(u))\}$
$= \beta^{3n}(\{x * y, \alpha(z), \alpha(u)\})$.
Therefore

$$\begin{aligned}
\circlearrowleft_{(x,y,z)} & \{x *_\beta y, (\beta^n\alpha)(z), (\beta^n\alpha)(u)\}_\beta \\
&= \circlearrowleft_{(x,y,z)}[\beta^{3n}(\{x * y, \alpha(z), \alpha(u)\}) \\
&= \beta^{3n}(\circlearrowleft_{(x,y,z)}\{x * y, \alpha(z), \alpha(u)\} \\
&= \beta^{3n}(0) \text{ (by (HLY6) for } A_\alpha) \\
&= 0
\end{aligned}$$

so that we get (HLY6) for A_{β^n}. Further, using (HLY7) for A_α and condition $\alpha\beta = \beta\alpha$, we compute
$\{(\beta^n\alpha)(x), (\beta^n\alpha)(y), u \; *_\beta \; v\}_\beta = \beta^{3n}(\{\alpha(x), \alpha(y), u * v\})$
$= \beta^{3n}(\{x, y, u\} * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}) = \beta^n(\beta^{2n}(\{x, y, u\}) * (\beta^{2n}\alpha^2)(v))$

$+\beta^n((\beta^{2n}\alpha^2)(u) * \beta^{2n}(\{x,y,v\})) = \{x,y,u\}_\beta \ *_\beta \ (\beta^{2n}\alpha^2)(v)$
$+(\beta^{2n}\alpha^2)(u) \ *_\beta \ \{x,y,v\}_\beta$
$= \{x,y,u\}_\beta \ *_\beta \ (\beta^n\alpha)^2(v) + (\beta^n\alpha)^2(u) \ *_\beta \ \{x,y,v\}_\beta.$
Thus (HLY7) holds for A_{β^n}. Using repeatedly the condition $\alpha\beta = \beta\alpha$ and the identity (HLY8) for A_α, the verification of (HLY8) for A_{β^n} is as follows.

$$\begin{aligned}
& \{(\beta^n\alpha)^2(x), (\beta^n\alpha)^2(y), \{u,v,w\}_\beta\}_\beta \\
= \ & \{(\beta^{2n}\alpha^2)(x), (\beta^{2n}\alpha^2)(y), \{u,v,w\}_\beta\}_\beta \\
= \ & \beta^{2n}(\{(\beta^{2n}\alpha^2)(x), (\beta^{2n}\alpha^2)(y), \beta^{2n}(\{u,v,w\})\}) \\
= \ & \beta^{4n}(\{\alpha^2(x), \alpha^2(y), \{u,v,w\}\}) \\
= \ & \beta^{4n}(\{\alpha^2(u)), \alpha^2(v)), \{x,y,w\}\}) \\
+ \ & \beta^{4n}(\{\{x,y,u\}, \alpha^2(v), \alpha^2(w)\}) \\
+ \ & \beta^{4n}(\{\alpha^2(u), \{x,y,v\}, \alpha^2(w)\} \\
= \ & \beta^{2n}(\{(\beta^{2n}\alpha^2)(u), (\beta^{2n}\alpha^2)(v), \beta^{2n}(\{x,y,w\})\}) \\
+ \ & \beta^{2n}(\{\beta^{2n}(\{x,y,u\}), (\beta^{2n}\alpha^2)(v), (\beta^{2n}\alpha^2)(w)\}) \\
+ \ & \beta^{2n}(\{(\beta^{2n}\alpha^2)(u), \beta^{2n}(\{x,y,v\}), (\beta^{2n}\alpha^2)(w)\}) \\
= \ & \{(\beta^n\alpha)^2(u), (\beta^n\alpha)^2(v), \{x,y,w\}_\beta\}_\beta \\
+ \ & \{\{x,y,u\}_\beta, (\beta^n\alpha)^2(v), (\beta^n\alpha)^2(w)\}_\beta \\
+ \ & \{(\beta^n\alpha)^2(u), \{x,y,v\}_\beta, (\beta^n\alpha)^2(w)\}_\beta.
\end{aligned}$$

Thus (HLY8) holds for A_{β^n}. Therefore, we get that A_{β^n} is a Hom-LY algebra. This finishes the proof. □

From Theorem 3.1 we have the following construction method of Hom-LY algebras from LY algebras (this yields examples of Hom-LY algebras). This method is an extension to binary-ternary algebras of a result due to D. Yau ([23], Theorem 2.3), giving a general construction method of Hom-algebras from their corresponding untwisted algebras. Such an extension to binary-ternary algebras is first mentioned in [6], Corollary 4.5.

Corollary 3.2. *Let* $(A, *, [,,])$ *be a LY algebra and* β *an endomorphism of* $(A, *, [,,])$*. If define on* A *a binary operation "*$\tilde{*}$*" and a ternary operation "*$\{,,\}$*" by*
$x \tilde{*} y := \beta(x * y)$,
$\{x,y,z\} := \beta^2([x,y,z])$,
then $(A, \tilde{*}, \{,,\}, \beta)$ *is a Hom-LY algebra.*

Proof. The proof follows if observe that Corollary 3.2 is Theorem 3.1 when $\alpha = Id$ and $n = 1$. □

A *Malcev algebra* [16] is an anticommutative algebra $(A, *)$ such that the *Malcev identity*

$$J(x, y, x * z) = J(x, y, z) * x$$

holds for all x, y, z in A, where $J(u, v, w) := \circlearrowleft_{(u,v,w)} (u * v) * w$ in $(A, *)$.

Corollary 3.3. *Let $(A, *)$ be a Malcev algebra and β any endomorphism of $(A, *)$. Define on A the operations*
$x \tilde{*} y := \beta(x * y)$,
$\{x, y, z\} := \beta^2((x * y) * z - (y * z) * x - (z * x) * y)$.
Then $(A, \tilde{*}, \{,,\}, \beta)$ is a Hom-LY algebra.

Proof. If consider on A the ternary operation $[x, y, z] := (x * y) * z - (y * z) * x - (z * x) * y$, $\forall x, y, z \in A$, then $(A, *, [,,])$ is a LY algebra [20]. Moreover, since β is an endomorphism of $(A, *)$, we have $\beta([x, y, z]) = (\beta(x) * \beta(y)) * \beta(z) - (\beta(y) * \beta(z)) * \beta(x) - (\beta(z) * \beta(x)) * \beta(y) = [\beta(x), \beta(y), \beta(z)]$ so that β is also an endomorphism of $(A, *, [,,])$. Then Corollary 3.2 implies that $(A, \tilde{*}, \{,,\}, \beta)$ is a Hom-LY algebra. □

4 Examples of 2-dimensional Hom-Lie-Yamaguti algebras

In this section, algebras are considered over the ground field of real numbers. Using the classification of real 2-dimensional LY algebras ([5], Corollary 2.7), we classify all the algebra morphisms of each of the proper (i.e. nontrivial) 2-dimensional LY algebras and then we obtain their corresponding 2-dimensional Hom-LY algebras (hence we give some application of Corollary 3.2).

For completeness, we recall that any 2-dimensional real LY algebra is isomorphic to one of the algebras (with basis $\{u, v\}$) of the following types:
(T1) $u * v = 0$, $[u, v, u] = \lambda u + \mu v$, $[u, v, v] = \gamma u - \lambda v$;
(T2) $u * v = u$, $[u, v, u] = 0$, $[u, v, v] = ku$;
(T3) $u * v = u + v$, $[u, v, u] = 0$, $[u, v, v] = 0$;
(T4) $u * v = au + bv$, $[u, v, u] = eu + fv$, $[u, v, v] = ku - ev$

with $a \neq 0$, $b \neq 0$, $e \neq 0$, $f \neq 0$, $k \neq 0$, and $af - be = 0 = bk + ae$. Also recall that a classification of complex 2-dimensional LY algebras is given in [21].

The algebras of type (T1) are either the zero algebra or nonzero Lie triple systems (a classification of real 2-dimensional Lie triple systems is given in [10]; see also [3], a classification of complex 2-dimensional Lie triple systems is found in [19]). According to Corollary 3.6 in [25], each of these Lie triple systems with a given endomorphism gives rise to a (real 2-dimensional) Hom-Lie triple system (i.e. a Hom-LY algebra with zero binary operation).

The algebras of type (T3) are real nonzero 2-dimensional Lie algebras (in fact (T3) is the only one, up to isomorphism, real nonzero 2-dimensional Lie algebra). Given any endomorphism of (T3) we get, by Theorem 3.3 of [23], its corresponding (real 2-dimensional) Hom-Lie algebra (i.e. a Hom-LY algebra with zero ternary operation).

Since Hom-Lie algebras as well as Hom-Lie triple systems are particular instances of Hom-LY algebras, we shall focus on algebras of types (T2) and (T4) in order to get nontrivial applications of Corollary 3.2.

In the LY algebra of type (T4), consider the basis change $\tilde{u} = au + bv$, $\tilde{v} = v$. Then some few transformations and the use of conditions $a \neq 0$, and $af - be = 0 = bk + ae$ imply that (T4) is isomorphic to

(T4') $\tilde{u} * \tilde{v} = a\tilde{u}$, $[\tilde{u}, \tilde{v}, \tilde{u}] = 0$, $[\tilde{u}, \tilde{v}, \tilde{v}] = k\tilde{u}$.

One observes that the algebra (T4') is isomorphic to the algebra (T2) by an isomorphism θ given by $\theta(u) = \alpha\tilde{u} + \beta\tilde{v}$, $\theta(v) = \gamma\tilde{u} + \delta\tilde{v}$ if and only if $a = \pm 1$ and $\alpha \neq 0$. Then the isomorphisms are

$\theta(u) = \alpha\tilde{u}$, $\theta(v) = \gamma\tilde{u} + v$ (when $a = 1$)

and

$\theta(u) = \alpha\tilde{u}$, $\theta(v) = \gamma\tilde{u} - v$ (when $a = -1$),

where $\alpha \neq 0$ and γ are real numbers.

Therefore, for a nontrivial application of Corollary 3.2, we shall consider algebras of type (T2) and type (T4') when $a \neq \pm 1$.

In general, if $(A, *, [,,])$ is a LY algebra, then a linear self-map ϕ of A is an endomorphism of $(A, *, [,,])$ if

$$\begin{cases} \phi(x * y) = \phi(x) * \phi(y) \\ \phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)], \end{cases} \tag{4.1}$$

for all $x, y, z \in A$. In order to determine ϕ, it suffices to apply the conditions (4.1) to the basis elements of A. Using elementary algebra, we are led to solve the resulting n simultaneous equations with respect to the coefficients expressing ϕ in the given basis (the number n depends on the dimension of A). These equations are not difficult to solve when $dimA = 2$. Suppose now that $dimA = 2$ and let $\{u, v\}$ be a basis of A. Then a linear self-map ϕ of A is given by a 2×2-matrix with respect to $\{u, v\}$. Set $\phi(u) = \epsilon u + \beta v$, $\phi(v) = \gamma u + \delta v$.

$\star$ Case of type (T2):

Then (4.1) induces the following simultaneous equations:

$$\begin{cases} \beta = 0 \\ \epsilon(1-\delta) = 0, \end{cases}$$

$$\begin{cases} \beta = 0 \\ \epsilon(1-\delta^2) = 0. \end{cases}$$

Resolving these simultaneous equations, we are led to the following endomorphisms ϕ of (T2):

$$I. \begin{cases} \phi(u) = 0 \\ \phi(v) = \gamma u + \delta v, \end{cases}$$

$$II. \begin{cases} \phi(u) = \epsilon u \\ \phi(v) = \gamma u + v, \end{cases}$$

where γ, δ, and $\epsilon \neq 0$ are any real numbers.

For each endomorphism I. and II. above, we apply Corollary 3.2 to find the corresponding Hom-LY algebra.

-For an endomorphism of type I., Corollary 3.2 implies that the LY algebra (T2) is twisted into the zero Hom-LY algebra by such an endomorphism.

-For an endomorphism of type II., Corollary 3.2 induces from (T2) the following Hom-LY algebras:

$$u \tilde{*} v = \epsilon u, \ \{u, v, u\} = 0, \ \{u, v, v\} = k\epsilon^2 u, \tag{4.2}$$

where $\epsilon \neq 0$ and $k \neq 0$ are real numbers.

Remark: *A LY algebra of type (T2) admits a nonzero twisting (i.e. the associated Hom-LY algebra is nonzero) if and only if $\epsilon \neq 0$, in which case the twisting maps are algebra automorphisms (including the identity map) of (T2).*

$\star$ Case of type (T4') (i.e. (T4)):

In this case, (4.1) leads to the same types I. and II. of endomorphisms as above. Therefore:

-for an endomorphism of type I., we get the zero Hom-LY algebra from (T4') by Corollary 3.2;

-for an endomorphism of type II., the application of Corollary 3.2 to (T4') gives the following Hom-LY algebras:

$$\tilde{u}\tilde{*}\tilde{v} = a\epsilon\tilde{u},\ \{\tilde{u},\tilde{v},\tilde{u}\} = 0,\ \{\tilde{u},\tilde{v},\tilde{v}\} = k\epsilon^2\tilde{u}, \tag{4.3}$$

where $\epsilon \neq 0$, $a \neq 0, \pm 1$ and $k \neq 0$ are real numbers.

Remark: *One observes that a Hom-LY algebra of type (4.2) is isomorphic to the one of type (4.3) by an isomorphism* $\theta(u) = \alpha\tilde{u} + \beta\tilde{v}$, $\theta(v) = \gamma\tilde{u} + \delta\tilde{v}$ *if and only if* $a = \pm 1$ *and* $\alpha \neq 0$. *Then the isomorphisms are*

$\theta(u) = \alpha\tilde{u}$, $\theta(v) = \gamma\tilde{u} + \tilde{v}$ *(when* $a = 1$*)*

and

$\theta(u) = \alpha\tilde{u}$, $\theta(v) = \gamma\tilde{u} - \tilde{v}$ *(when* $a = -1$*).*

Therefore the Hom-LY algebras (4.2) and (4.3) are isomorphic if and only if the LY algebras (T2) and (T4') are isomorphic.

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