A new approach for regularization of inverse problems in image processing

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\textit{10\textsuperscript{th} African Conference on Research in Computer Science and Applied Mathematics - CARI 2010}
October 18 - 21, 2010, Yamoussoukro, Côte d'Ivoire
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Ingredients

**Physical system**

\[ \mathcal{M} : \mathcal{V} \rightarrow \mathcal{Y} \]

\[ \mathbf{v} \mapsto \mathbf{y} = \mathcal{M}(\mathbf{v}) \]

- \( \mathbf{y} \in \mathcal{Y} \), the system state
- \( \mathbf{v} \in \mathcal{V} \), the control variable
- \( \mathcal{M} \), model mapping \( \mathcal{V} \) to \( \mathcal{Y} \)

**Observation**

\( \mathbf{y}^o \)

- \( \mathbf{y}^o \in \mathcal{O} \), observed state

**Observation system**

\[ \mathcal{H} : \mathcal{Y} \rightarrow \mathcal{O} \]

\[ \mathbf{y} \mapsto \mathcal{H}(\mathbf{y}) \]

- \( \mathcal{H} \) observation operator mapping \( \mathcal{Y} \) to \( \mathcal{O} \)
Inverse problems: variational formulation

Generalized Diffusion regularization

Application

Conclusion

Definition

A priori knowledges

Regularization

**Definition**

Giving observed state \( y^o \),

**Inverse problem (unconstrained)**

Find \( \mathbf{v}^* = \text{MinArg}(J(\mathbf{v})), \mathbf{v} \in \mathcal{V} \) where

\[
J(\mathbf{v}) = J_0(\mathbf{v}) = \frac{1}{2} \| \mathcal{H}(\mathcal{M}(\mathbf{v})) - y^o \|^2_0
\]  

(1)

under adequate conditions, the solution \( \mathbf{v}^* \) is given by the Euler-Lagrange Equation \( \nabla J(\mathbf{v}^*) = 0 \)

**Problems**

- ill-posedness \( \Rightarrow \) use a priori knowledges;
- ill-conditionning \( \Rightarrow \) use preconditioning.
A priori knowledges

For a priori knowledge $\mathcal{A}$, set $J = J_0 + J_\mathcal{A}$ where $J_\mathcal{A}$ is defined to force the solution to satisfy $\mathcal{A}$

### Use of a priori informations

- **Background** $\mathbf{v}^b$ and background error covariance $\mathbf{B}$

  $$J_b = \frac{1}{2} \alpha_b \| \mathbf{v} - \mathbf{v}^b \|^2_{\mathbf{B}^{-1}} \quad (2)$$

- **Regularity of the solution**: $\Phi$-smooth (minimum gradient)

  $$J_r = \frac{1}{2} \alpha_r \| \Phi(\mathbf{v}) \|^2 \quad (3)$$

  $\Phi$ function of the derivatives of $\mathbf{v}$
Vector fields regularization

**first order regularization:** first order derivatives of $\mathbf{v}$

$$\Phi^{(1)} \left( \frac{\partial v_i}{\partial x_j} \right)_{1 \leq i, j \leq n},$$

- gradient penalization: $J_{\nabla}(\mathbf{v}) = \frac{1}{2} \alpha_{\nabla} \int_{\Omega} \sum_{i=1}^{n} \|\nabla v_i\|^2 d\mathbf{x}$

**second order regularization:** second order derivatives of $\mathbf{v}$

$$\Phi^{(2)} \left( \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right)_{1 \leq i, j, k \leq n},$$

- Suter regularization:
  $$J_{suter}(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \alpha_{\nabla \text{div}} \|\nabla \text{div}(\mathbf{v})\|^2 + \alpha_{\nabla \text{curl}} \|\nabla \text{curl}(\mathbf{v})\|^2 d\mathbf{x}$$

$\Rightarrow$ difficult to defined optimal weighting parameter(s)

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regularization for inverse problem
Notations and definition

Let:
- \( \mathbf{v}(\mathbf{x}) \) be an incomplete/inconsistent control variable, with \( \mathbf{x} \in \Omega \) the physical space
- \( \Phi(\mathbf{v}) \) regularization operator as defined previously
- \( \varphi(\mathbf{x}) \) a scalar positive trust function given the quality of \( \mathbf{v} \) at \( \mathbf{x} \)

\[
\begin{cases}
\text{small value meaning bad/lack/inconsistent control variable} \\
\text{large value for good quality control variable}
\end{cases}
\]

we define restored control variable \( \mathbf{u}^* = \text{MinArg}(\varepsilon(\mathbf{u})) \), \( \mathbf{u} \in \mathcal{V} \)

\[
\varepsilon(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \| \Phi(\mathbf{u}(\mathbf{x})) \|^2 + \varphi(\mathbf{x}) \| \mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x}) \|^2 d\mathbf{x} \tag{4}
\]
\[ \varepsilon(u) = \frac{1}{2} \int_{\Omega} \| \Phi(u(x)) \|^2 + \varphi(x) \| u(x) - v(x) \| \, dx \]

\( \varepsilon \) is minimized by setting \( u \) to be:
- close to \( v \) when \( \varphi \) is large (\( v \) has adequate properties)
- \( \Phi \) - regular when \( \varphi \) is small (otherwise)

Under adequate conditions MinArg(\( \varepsilon \)) is given by the Euler-Lagrange condition

\[ \nabla_u \varepsilon(u) = 0 \] (5)

Gateaux derivatives development leads to

\[ \nabla_u \varepsilon(u) = \Phi^* \circ \Phi(u(x)) + \varphi(x)(u(x) - v(x)) \] (6)
Gradient penalization: mathematical expression

\[ J_\nabla(v) = \frac{1}{2} \alpha_\nabla \int_\Omega \sum_{i=1}^{n} \| \nabla v_i \|^2 dx \]

Applied as smoothing operator, we get

\[ \Phi^*_\nabla \circ \Phi_\nabla = -\Delta, \text{ with boundary conditions: } \nabla u_i \perp \nu \text{ on } \partial \Omega \]

\[ \Rightarrow \nabla \varepsilon_\nabla(u_i) = -\Delta u_i(x) + \varphi(x)(u_i(x) - v_i(x)), 1 \leq i \leq n \quad (7) \]
Numerical implementation

Classical implementation: given $\nabla \varepsilon$, use descent-type algorithms.

Problem: solve the Euler-Lagrange equation

$$\Delta u_i - \varphi(x)(u_i(x) - v_i(x)) = 0, \quad 1 \leq i \leq n \quad (8)$$

considers $u_i$ as a function of time and solve the equivalent problem

$$\frac{\partial}{\partial t} u_i(x, t) = \Delta u_i(x, t) - \varphi(x)(u_i(x, t) - v_i(x)), \quad 1 \leq i \leq n \quad (9)$$

known as the generalized diffusion equations.

As diffusion operator, it can directly be used in background covariance [see Weaver et al.]
optical flow of Horn and Shunck: luminance conservation

\[ \frac{df}{dt} = 0 \]  

(10)

\( f(x, t) \) noted \( f \) is the luminance function.

For geophysical fluid images, the mass conservation equation is more adequate [Fitzpatrick 1985]

\[ \frac{df}{dt} + f(\nabla \cdot \mathbf{v}) = 0 \]  

(11)

\( \mathbf{v}(x) \) is the velocity at \( x \)

given the luminance function \( f(x, 0) = f^0(x) \) at time 0, solution to equations (10) or (11) defines \( f(x, t) \) as function of the static velocity field \( \mathbf{v}(x) \)

\[ \mathcal{M} : \mathcal{V} \rightarrow \mathcal{F} \]

\[ \mathbf{v} \mapsto f = \mathcal{M}(\mathbf{v}) \]
motion estimation is reduced to the inverse problem:

$$\delta \mathbf{v}^* = \operatorname{MinArg}(J(\delta \mathbf{v}))$$

with

$$J(\delta \mathbf{v}) = \frac{1}{2} \| M(\mathbf{v}^b + \delta \mathbf{v}) - \mathbf{f}^o \|^2_F + \frac{1}{2} \| \delta \mathbf{v} \|^2_V$$  \hspace{1cm} (12)

**Aperture problem:**

only motion along the normal to iso-contours can be inferred $\Rightarrow$

use regularization
Proposition: define trust function $\varphi$

to have large values on discontinuities (contours) for motion component along the normal to the contour, and small values in homogeneous areas.

Example: set $\varphi$ to be the contours map $c^1$ or $c^2$ defined as

$$c^1(x, f) = \| \nabla_x f(x) \|^2$$
$$c^2(x, f) = \| \nabla_x (G_{\sigma}(x) * f(x)) \|^2$$
Twin experiments

Direct Image sequences assimilation [Titaud et al 2009] ⇒ true initial state (velocity fields)

Images from [J.-B. Flór (LEGI) and I. Eames, 2002]
Error analysis: Tikhonov regularization

Evolution of diagnostic functions with respect to the weighting parameter $\alpha$

![Diagnostics at convergence - Tikhonov regularization](image)

- Normalized angular error
- Normalized vorticity error
- Normalized velocity error
- Normalized observation cost function
- Normalized cost function

Mean values

Max values

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Regularization for inverse problems
Error analysis: gradient penalization

Evolution of diagnostic functions with respect to the weighting parameter $\alpha$ and $\nabla$.

Diagnostics at convergence - gradient penalization

Mean values and max values for different diagnostic functions:
- Normalized angular error
- Normalized vorticity error
- Normalized velocity error
- Normalized obs cost function
- Normalized cost function

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Error analysis: Generalised diffusion

Evolution of diagnostic functions with respect to the weighting parameter \( \alpha_{GD} \)

Diagnostics at convergence - GD, homo + id

Weighting parameter

mean values

max values

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regularization for inverse problem
Error analysis: Comparison - cost function

Evolution of the observation cost function with minimization iterations

Optimal weighting parameter

![Graph showing the evolution of the observation cost function with iterations, comparing different methods such as GD, homogeneous smoothing, id-weighting, flow-driven isotropic, etc.]

- GD, homogeneous smoothing, id-weighting
- Flow-driven isotropic
- Image-driven isotropic
- Second order div-curl penalization
- Gradient penalization
- No regularization
Error analysis: Comparison - velocity error

Evolution of the velocity error with minimization iterations

Optimal weighting parameter

Velocity, mean error

- GD, homogeneous smoothing, id-weighting
- Flow-driven isotropic
- Image-driven isotropic
- Second order div-curl penalization
- Gradient penalization
- No regularization

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Regularization for inverse problem
Error analysis: Comparison - vorticity error

Evolution of the vorticity error with minimization iterations

Optimal weighting parameter

Vorticity, mean error vs iterations count

Mean values

Max values

Graph showing the evolution of vorticity error with iterations for different weighting parameters.
Error analysis: Comparison - angular error

Evolution of the angular error with minimization iterations

Optimal weighting parameter

Mean angular error vs. Iterations count for mean values and max values.
Analysis: vector field

true

Tikhonov regularization

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regularization for inverse problem
Motion estimation problem

From regularization to pseudo covariance operator

Experimental result

Analysis: vector field

true
gradient penalization
Motion estimation problem
From regularization to pseudo covariance operator
Experimental result

Analysis: vector field

true

Generalised Diffusion
Conclusion

Inverse problems:
- ill-posed $\Rightarrow$ use regularization
- ill-conditioned $\Rightarrow$ use preconditioner

proposed: promising approach for regularization of inverse problems.